Supplementary Notes for Jack Cowan's Lectures
The Fokker-Planck Equation

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Consider again the random walk defined by $X_t(\Delta t) = Z_0 + Z_{\Delta t} + Z_{2\Delta t} + \cdots + Z_{n\Delta t}$ where $t = n\Delta t$ and

$$Z_i = \begin{cases} \Delta x \text{ with probability } p \\ -\Delta x \text{ with probability } q = 1 - p \end{cases}$$

To simplify the notation we shall leave out the dependence on Δt in the following so $X_t = X_t(\Delta t)$ until further notice. Consider the conditional probability $Pr\{X_t = x | X_{t_0} = x_0\}$. There are two ways to get to position x at time t namely from position $x - \Delta x$ at time $t - \Delta t$ by moving $+\Delta x$ or from position $x + \Delta x$ by moving $-\Delta x$. Thus we have

$$\begin{aligned} &Pr\{X_t = x | X_{t_0} = x_0\} \\ &\quad = pPr\{X_{t-\Delta t} = x - \Delta x | X_{t_0} = x_0\} + qPr\{X_{t-\Delta t} = x + \Delta x | X_{t_0} = x_0\} \end{aligned}$$

Consider the conditional density $\phi_{X_t|X_{t_0}}(x|x_0) = \phi_{X_t,X_{t_0}}(x, x_0)/\phi_{X_{t_0}}(x_0)$. We have $Pr\{X_t = x + \Delta x | X_{t_0} = x_0\} - Pr\{X_t = x | X_{t_0} = x_0\} \sim \phi_{X_t|X_{t_0}}(x|x_0)\Delta x$ for Δx small, because

$$\begin{aligned} \phi_{X_t|X_{t_0}}(x|x_0) &= \frac{d}{dx} Pr\{X_t = x | X_{t_0} = x_0\} \\ &= \lim_{\Delta x \to 0} \frac{Pr\{X_t = x + \Delta x | X_{t_0} = x_0\} - Pr\{X_t = x | X_{t_0} = x_0\}}{\Delta x} \end{aligned}$$

We write $\phi(x, t; x_0, t_0)$ for $\phi_{X_t|X_{t_0}}(x|x_0)$. Then we have

$$\begin{aligned} \phi(x, t; x_0, t_0)\Delta x &= Pr\{X_t = x + \Delta x | X_{t_0} = x_0\} - Pr\{X_t = x | X_{t_0} = x_0\} \\ &= pPr\{X_{t-\Delta t} = x | X_{t_0} = x_0\} + qPr\{X_{t-\Delta t} = x + 2\Delta x | X_0 = x_0\} \\ &- (pPr\{X_{t-\Delta t} = x - \Delta x | X_{t_0} = x_0\} + qPr\{X_{t-\Delta} = x + \Delta x | X_{t_0} = x_0\} \\ &= p(Pr\{X_{t-\Delta t} = x | X_{t_0} = x_0\} - Pr\{X_{t-\Delta t} = x - \Delta x | X_{t_0} = x_0\}) \\ &- q(Pr\{X_{t-\Delta t} = x + 2\Delta x | X_{t_0} = x_0\} - Pr\{X_{t-\Delta t} = x + \Delta x | X_{t_0} = x_0\}) \\ &= p\phi(x, t - \Delta t; x_0, t_0)\Delta x + q\phi(x + \Delta x, t - \Delta t; x_0, t_0)\Delta x \end{aligned}$$

Hence cancelling Δx we get the formula

$$\phi(x,t;x_0,t_0) = p\phi(x-\Delta x, t-\Delta t; x_0,t_0) - q\phi(x+\Delta x, t-\Delta t; x_0, t_0)$$

Recall the Taylor series expansion of a function of two variables:

$$\psi(x+\Delta x, t+\Delta t) =$$
$$\psi(x,t)+\Delta x\frac{\partial\psi}{\partial x}+\Delta t\frac{\partial\psi}{\partial t}+\frac{1}{2}(\Delta x)^2\frac{\partial^2\psi}{\partial x^2}+\frac{1}{2}\Delta x\Delta t\frac{\partial^2\psi}{\partial x\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2\psi}{\partial t^2}+\cdots+\text{terms of higher order}$$

Now substituting this into our previous equation we get

$$\phi(x,t;x_0,t_0)$$
$$= p(\phi(x,t;x_0,t_0)-\Delta x\frac{\partial\phi}{\partial x}-\Delta t\frac{\partial\phi}{\partial t}+\frac{1}{2}(\Delta x)^2\frac{\partial^2\phi}{\partial x^2}+\frac{1}{2}\Delta x\Delta t\frac{\partial^2\phi}{\partial x\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2\phi}{\partial t^2}\dots)$$
$$+q(\phi(x,t;x_0,t_0)+\Delta x\frac{\partial\phi}{\partial x}-\Delta t\frac{\partial\phi}{\partial t}+\frac{1}{2}(\Delta x)^2\frac{\partial^2\phi}{\partial x^2}-\frac{1}{2}\Delta x\Delta t\frac{\partial^2\phi}{\partial x\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2\phi}{\partial t^2}\dots)$$

Using $p+q=1$ we get

$$0 = -(p-q)\Delta x\frac{\partial\phi}{\partial x}-\Delta t\frac{\partial\phi}{\partial t}+\frac{1}{2}(\Delta x)^2\frac{\partial^2\phi}{\partial x^2}+(p-q)\frac{1}{2}\Delta x\Delta t\frac{\partial^2\phi}{\partial x\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2\phi}{\partial t^2}\dots$$

Substituting the expressions for $p, q, \Delta x$: $p = \frac{1}{2}(1+\frac{\mu\sqrt{\Delta t}}{\sigma})$, $q = \frac{1}{2}(1-\frac{\mu\sqrt{\Delta t}}{\sigma})$ and $\Delta x = \sigma\sqrt{\Delta t}$, we get

$$0 = -\frac{\mu\sqrt{\Delta t}}{\sigma}\sigma\sqrt{\Delta t}\frac{\partial\phi}{\partial x}-\Delta t\frac{\partial\phi}{\partial t}+\frac{1}{2}(\sigma\sqrt{\Delta t})^2\frac{\partial^2\phi}{\partial x^2}+(p-q)\frac{1}{2}\Delta x\Delta t\frac{\partial^2\phi}{\partial x\partial t}+\frac{1}{2}(\Delta t)^2\frac{\partial^2\phi}{\partial t^2}\dots$$

Dividing through by Δt and letting $\Delta t \to 0$ we get the partial differential equation:

$$0 = -\mu\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2}$$

This is the Fokker-Planck (also called the *forward*) equation and the conditional density function $\phi_{X_t|X_{t_0}}(x|x_0)$ is a solution.

This partial differential equation can be solved using the Fourier transform. Assume for simplicity that $t=0$. We first change variables $y = x - x_0 - \mu t$, $s = \sigma^2 t$. By the chain rule we have $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial\phi}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial\phi}{\partial y}$ and $\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\phi}{\partial s}\frac{\partial s}{\partial t} = -\mu\frac{\partial\phi}{\partial y} + \sigma^2\frac{\partial\phi}{\partial s}$.

Substituting into the partial differential equation we get

$$-\mu\frac{\phi}{\partial y} + \sigma^2\frac{\partial\phi}{\partial s} = \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial y^2} - \mu\frac{\partial\phi}{\partial y}$$

or

$$\frac{\partial \phi}{\partial s} = \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}$$

The Fourier Transform

Let f be a function. The Fourier transform of f is given by $Ff(u) = \int_{-\infty}^{\infty} f(x)e^{iux}dx$. Recall the integration by parts formula: $\int f'g = fg - \int fg'$. This formula is an immediate consequence of the product rule for differentiation, $(fg)' = f'g + fg'$. We shall assume that $f(x) \to 0$ for $x \to \pm\infty$. Consider $Ff'(u) = \int_{-\infty}^{\infty} f'(x)e^{iux}dx = f(x)e^{iux}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(e^{iux})'dx$. The term $f(x)e^{iux}|_{-\infty}^{-\infty}$ is 0 since $f(\pm\infty) = 0$ and $|e^{iux}| = 1$. The integral equals $iu\int_{-\infty}^{\infty} f(x)e^{iux}dx = iuFf(u)$. Thus we obtain the formula $Ff'(u) = 0 - iuFf(u) = -iuFf(u)$.

Since we are looking for a solution that looks like a probability distribution we want ϕ and all its derivatives to vanish at $y = \pm\infty$.

Now let $M(u,s) = \int_{-\infty}^{\infty} \phi(y,s)e^{iuy}dy$, the Fourier transform of ϕ viewed as a function in y. Differentiating under the integral sign we get $\frac{\partial M}{\partial s} = \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial s}(y,s)e^{iuy}dy = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2}(y,s)e^{iuy}dy = \frac{1}{2}F\frac{\partial^2 \phi}{\partial y^2}(u,s)$.

Using our formula above we have $F\frac{\partial^2 \phi}{\partial y^2}(u,s) = -iuF\frac{\partial \phi}{\partial y}(u,s) = (-iu)^2 F\phi(u,s) = -u^2 M(u,s)$. Hence M satisfies the differential equation $\frac{\partial M}{\partial s} = -\frac{1}{2}u^2 M(u,s)$. It follows that $M(u,s) = C(u)e^{-\frac{1}{2}u^2 s}$. In order to specify the function $C(u)$ we need a boundary condition at $s = 0$. So what is $\phi(y,0)$?. This is the density function of the distribution $Pr(X_0 - x_0 < c | X_0 = x_0) = \int_{-\infty}^{c} \phi(x - x_0, 0; x_0, 0)dx$. But clearly

$$Pr(X_0 - x_0 < c | X_0 = x_0) = \begin{cases} 0 \text{ if } c \leq 0 \\ 1 \text{ if } c > 0 \end{cases}$$

It follows that $\phi(x - x_0, 0; x_0, 0)$ (= derivative of the distribution) is 0 for $x \neq x_0$ and $\int_{-\infty}^{\infty} \phi(x - x_0, 0; x_0, 0)dx = 1$. Thus $\phi(x - x_0, 0; x_0, 0) = \delta(x - x_0)$, the Dirac delta function at x_0 (which of course is not a function at all) and so $\phi(y,0) = \delta(y)$. For any function f we have $\int_{-\infty}^{\infty} f(y)\delta(y)dy = f(0)$ and so we get $M(u,0) = \int_{-\infty}^{\infty} \phi(y,0)e^{iuy}dy = \int_{-\infty}^{\infty} \delta(y)e^{iuy}dy = e^{iu0} = 1$. Hence $M(u,0) = \int_{-\infty}^{\infty} \phi(y,0)dy = 1$ so $C(u)$ is identically 1 and $M(u,s) = e^{-\frac{1}{2}u^2 s}$.

(To prove $\int_{-\infty}^{\infty} f(y)\delta(y)dy = f(0)$ we could proceed as follows: look at δ as the limit of the sequence of functions $\{\delta_n\}$, where δ_n is the function which takes the value n on the interval $[0, \frac{1}{n}]$ and is 0 outside this interval. Clearly $\int_{-\infty}^{\infty} \delta_n(x)dx = 1$. $\int_{-\infty}^{\infty} \delta(y)f(y)dy = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(y)\delta_n(y)dy$. To compute this integral let $F(t) = \int_{-\infty}^{t} f(y)dy$, then $F'(x) = f(x)$ and we have

$\int_{-\infty}^{\infty} f(y)\delta_n(y)dy = n\int_0^{\frac{1}{n}} F(x)dx = n(F(\frac{1}{n}) - F(0))$. Hence $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(y)\delta_n(y)dy = \lim_{n\to\infty} \frac{F(\frac{1}{n}) - F(0)}{\frac{1}{n}} = F'(0) = f(0).)$

Consider now the the gaussian density with mean 0 and variance s: $f(x) = \frac{1}{\sqrt{2\pi s}} e^{\frac{-x^2}{2s}}$.

We have

$$\begin{aligned}
Ff(u) &= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2s}} e^{iu\cdot x} dx \\
&= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{\frac{-x^2 + 2siu\cdot x}{2s}} dx \\
&= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{(x - ius)^2 + u^2 s^2}{2s}} dx \\
&= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{(x - isu)^2}{2s}} e^{-\frac{u^2 s}{2}} dx = e^{-\frac{u^2 s}{2}}
\end{aligned}$$

Thus $Ff(u) = M(u, s)$ and so $\phi(y, s) = f(y) = \frac{1}{\sqrt{2\pi s}} e^{\frac{-y^2}{2s}}$. Substituting back in our expressions for y and s we end up with $\phi(x, t; x_0, 0) = \frac{1}{\sigma\sqrt{2\pi t}} e^{\frac{x - x_0 - \mu t}{2\sigma^2 t}}$ which is the density for a normal distribution with mean $x_0 + \mu t$ and variance $\sigma^2 t$.

In general when t_0 is not necessarily 0 we get

$$\phi(x, t; x_0, t_0) = \frac{1}{\sigma\sqrt{2\pi(t - t_0)}} e^{-\frac{x - x_0 - \mu(t - t_0)}{2\sigma^2(t - t_0)}}$$