Consider again the random walk defined by $X_t(\Delta t) = Z_0 + Z_{\Delta t} + Z_{2\Delta t} + \cdots + Z_{n\Delta t}$ where $t = n\Delta t$ and

$$Z_i = \begin{cases} \Delta x \text{ with probability } p \\ -\Delta x \text{ with probability } q = 1 - p \end{cases}$$

To simplify the notation we shall leave out the dependence on $\Delta t$ in the following so $X_t = X_t(\Delta t)$ until further notice. Consider the conditional probability $Pr\{X_t = x|X_{t_0} = x_0\}$. There are two ways to get to position $x$ at time $t$ namely from position $x - \Delta x$ at time $t - \Delta t$ by moving $+\Delta x$ or from position $x + \Delta x$ by moving $-\Delta x$. Thus we have

$$Pr\{X_t = x|X_{t_0} = x_0\} = pPr\{X_{t_\Delta t} = x - \Delta x|X_{t_0} = x_0\} + qPr\{X_{t_\Delta t} = x + \Delta x|X_{t_0} = x_0\}$$

Consider the conditional density $\phi_{X_t|X_{t_0}}(x|x_0) = \phi_{X_t,X_{t_0}}(x,x_0)/\phi_{X_{t_0}}(x_0)$. We have $Pr\{X_t = x + \Delta x|X_{t_0} = x_0\} - Pr\{X_t = x|X_{t_0} = x_0\} \sim \phi_{X_t|X_{t_0}}(x|x_0)\Delta x$ for $\Delta x$ small, because

$$\phi_{X_t|X_{t_0}}(x|x_0) = \frac{d}{dx}Pr\{X_t = x|X_{t_0} = x_0\} = \lim_{\Delta x \to 0} \frac{Pr\{X_t = x + \Delta x|X_{t_0} = x_0\} - Pr\{X_t = x|X_{t_0} = x_0\}}{\Delta x}$$

We write $\phi(x,t;x_0,t_0)$ for $\phi_{X_t|X_{t_0}}(x|x_0)$. Then we have

$$\phi(x,t;x_0,t_0)\Delta x = Pr\{X_t = x + \Delta x|X_{t_0} = x_0\} - Pr\{X_t = x|X_{t_0} = x_0\}
= pPr\{X_{t_\Delta t} = x|X_{t_0} = x_0\} + qPr\{X_{t_\Delta t} = x + 2\Delta x|X_{t_0} = x_0\}
- qPr\{X_{t_\Delta t} = x - \Delta x|X_{t_0} = x_0\} + qPr\{X_{t_\Delta t} = x + \Delta x|X_{t_0} = x_0\}
= p(Pr\{X_{t_\Delta t} = x|X_{t_0} = x_0\} - Pr\{X_{t_\Delta t} = x - \Delta x|X_{t_0} = x_0\})
- q(Pr\{X_{t_\Delta t} = x + 2\Delta x|X_{t_0} = x_0\} - Pr\{X_{t_\Delta t} = x + \Delta x|X_{t_0} = x_0\})
= p\phi(x,t - \Delta t;x_0,t_0)\Delta x + q\phi(x + \Delta x,t - \Delta t;x_0,t_0)\Delta x$$

October 25, 2002
Hence cancelling $\Delta x$ we get the formula

$$\phi(x, t; x_0, t_0) = p\phi(x - \Delta x, t - \Delta t; x_0, t_0) - q\phi(x + \Delta x, t - \Delta t; x_0, t_0)$$

Recall the Taylor series expansion of a function of two variables:

$$\psi(x + \Delta x, t + \Delta t) =$$

$$\psi(x, t) + \Delta x \frac{\partial \psi}{\partial x} + \Delta t \frac{\partial \psi}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} \Delta x \Delta t \frac{\partial^2 \psi}{\partial x \partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \psi}{\partial t^2} + \cdots + \text{terms of higher order}$$

Now substituting this into our previous equation we get

$$\phi(x, t; x_0, t_0)$$

$$= p(\phi(x, t; x_0, t_0) - \Delta x \frac{\partial \phi}{\partial x} - \Delta t \frac{\partial \phi}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \Delta x \Delta t \frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \phi}{\partial t^2} \cdots)$$

$$+ q(\phi(x, t; x_0, t_0) + \Delta x \frac{\partial \phi}{\partial x} - \Delta t \frac{\partial \phi}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{1}{2} \Delta x \Delta t \frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \phi}{\partial t^2} \cdots)$$

Using $p + q = 1$ we get

$$0 = -(p - q)\Delta x \frac{\partial \phi}{\partial x} - \Delta t \frac{\partial \phi}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 \phi}{\partial x^2} + (p - q) \frac{1}{2} \Delta x \Delta t \frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \phi}{\partial t^2} \cdots$$

Substituting the expressions for $p, q, \Delta x$: $p = \frac{1}{2} (1 + \frac{\mu \sqrt{\Delta t}}{\sigma})$, $q = \frac{1}{2} (1 - \frac{\mu \sqrt{\Delta t}}{\sigma})$ and $\Delta x = \sigma \sqrt{\Delta t}$, we get

$$0 = -\frac{\mu \sqrt{\Delta t}}{\sigma} \sigma \sqrt{\Delta t} \frac{\partial \phi}{\partial x} - \Delta t \frac{\partial \phi}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 \phi}{\partial x^2} + (p - q) \frac{1}{2} \Delta x \Delta t \frac{\partial^2 \phi}{\partial x \partial t} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 \phi}{\partial t^2} \cdots$$

Dividing through by $\Delta t$ and letting $\Delta t \to 0$ we get the partial differential equation:

$$0 = -\mu \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2}$$

This is the Fokker-Planck (also called the forward) equation and the conditional density function $\phi_{X_1|X_0}(x|x_0)$ is a solution.

This partial differential equation can be solved using the Fourier transform. Assume for simplicity that $t = 0$. We first change variables $y = x - x_0 - \mu t$, $s = \sigma^2 t$. By the chain rule we have $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial \phi}{\partial s} \frac{\partial s}{\partial x}$ and $\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial s} \frac{\partial s}{\partial t}$.

Substituting into the partial differential equation we get

$$-\mu \frac{\partial \phi}{\partial y} + \sigma^2 \frac{\partial^2 \phi}{\partial s^2} = \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial y^2} - \mu \frac{\partial \phi}{\partial y}$$
or
\[ \frac{\partial \phi}{\partial s} = \frac{1}{2} \frac{\partial^2 \phi}{\partial y^2} \]

**The Fourier Transform**

Let \( f \) be a function. The Fourier transform of \( f \) is given by \( Ff(u) = \int_{-\infty}^{\infty} f(x)e^{ixu}dx \). Recall the integration by parts formula: \( \int f'g = fg - \int fg' \). This formula is an immediate consequence of the product rule for differentiation, \((fg)' = f'g + fg'\). We shall assume that \( f(x) \to 0 \) for \( x \to \pm \infty \).

Consider \( Ff'(u) = \int_{-\infty}^{\infty} f'(x)e^{ixu}dx = f(x)e^{ixu}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(e^{ixu})'dx \). The term \( f(x)e^{ixu}|_{-\infty}^{\infty} \) is 0 since \( f(\pm \infty) = 0 \) and \( |e^{ixu}| = 1 \). The integral equals \( iu \int_{-\infty}^{\infty} f(x)e^{ixu}dx = iuFf(u) \). Thus we obtain the formula \( Ff'(u) = 0 - iuFf(u) = -iuFf(u) \).

Since we are looking for a solution that looks like a probability distribution we want \( \phi \) and all its derivatives to vanish at \( y = \pm \infty \).

Now let \( M(u,s) = \int_{-\infty}^{\infty} \phi(y,s)e^{iyu}dy \), the Fourier transform of \( \phi \) viewed as a function in \( y \). Differentiating under the integral sign we get \( \frac{\partial M}{\partial s} = \int_{-\infty}^{\infty} \frac{\partial \phi}{\partial s}(y,s)e^{iyu}dy = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2}(y,s)e^{iyu}dy = \frac{1}{2} F\frac{\partial^2 \phi}{\partial y^2}(u,s) \).

Using our formula above we have \( F\frac{\partial^2 \phi}{\partial y^2}(u,s) = -iuF\frac{\partial \phi}{\partial y}(u,s) = (-iu)^2 F\phi(u,s) = -u^2 M(u,s) \). Hence \( M \) satisfies the differential equation \( \frac{\partial M}{\partial s} = -\frac{1}{2} u^2 M(u,s) \).

It follows that \( M(u,s) = C(u)e^{-\frac{1}{2} u^2 s} \). In order to specify the function \( C(u) \) we need a boundary condition at \( s = 0 \). So what is \( \phi(y,0) \)? This is the density function of the distribution \( Pr(X_0 - x_0 < c | X_0 = x_0) = \int_{-\infty}^{c} \phi(x-x_0,0; x_0,0)dx \).

But clearly
\[ Pr(X_0 - x_0 < c | X_0 = x_0) = \begin{cases} 0 & \text{if } c \leq 0 \\ 1 & \text{if } c > 0 \end{cases} \]

It follows that \( \phi(x-x_0,0; x_0,0) \) (= derivative of the distribution) is 0 for \( x \neq x_0 \) and \( \int_{-\infty}^{\infty} \phi(x-x_0,0; x_0,0)dx = 1 \). Thus \( \phi(x-x_0,0; x_0,0) = \delta(x-x_0) \), the Dirac delta function at \( x_0 \) (which of course is not a function at all) and so \( \phi(y,0) = \delta(y) \). For any function \( f \) we have \( \int_{-\infty}^{\infty} f(y)\delta(y)dy = f(0) \) and so we get \( M(u,0) = \int_{-\infty}^{\infty} \phi(y,0)e^{iyu}dy = \int_{-\infty}^{\infty} \delta(y)e^{iyu}dy = e^{iu0} = 1 \). Hence
\[ M(u,0) = \int_{-\infty}^{\infty} \phi(y,0)dy = 1 \]
so \( C(u) \) is identically 1 and \( M(u,s) = e^{-\frac{1}{2} u^2 s} \).

(To prove \( \int_{-\infty}^{\infty} f(y)\delta(y)dy = f(0) \) we could proceed as follows: look at \( \delta \) as the limit of the sequence of functions \( \{\delta_n\} \), where \( \delta_n \) is the function which takes the value \( n \) on the interval \([0, \frac{1}{n}]\) and is 0 outside this interval. Clearly \( \int_{-\infty}^{\infty} \delta_n(x)dx = 1 \). \( \int_{-\infty}^{\infty} \delta(x)f(x)dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} f(y)\delta_n(y)dy \). To compute this integral let \( F(t) = \int_{-\infty}^{t} f(y)dy \), then \( F'(x) = f(x) \) and we have
\[
\int_{-\infty}^{\infty} f(y) \delta_n(y) \, dy = n \int_0^1 F(x) \, dx = n \left( F\left( \frac{1}{n} \right) - F(0) \right). \text{ Hence } \lim_{n \to \infty} \int_{-\infty}^{\infty} f(y) \delta_n(y) \, dy = \\
\lim_{n \to \infty} \frac{F\left( \frac{1}{n} \right) - F(0)}{\frac{1}{n}} = F'(0) = f(0). \]

Consider now the Gaussian density with mean 0 and variance \( s \):
\[
f(x) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}.
\]
We have
\[
Ff(u) = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2s}} e^{iu \cdot x} \, dx
\]
\[
= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} \frac{x^2 - 2uisu \cdot x}{2s} \, dx
\]
\[
= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} \frac{(x - ius)^2 + u^2s^2}{2s} \, dx
\]
\[
= \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{\infty} e^{-\frac{(x - ius)^2}{2s}} - \frac{u^2s}{2} \, dx = e^{-\frac{u^2s}{2}}
\]
Thus \( Ff(u) = M(u, s) \) and so \( \phi(y, s) = f(y) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{y^2}{2s}} \).

Substituting back in our expressions for \( y \) and \( s \) we end up with
\[
\phi(x; t; x_0, 0) = \frac{1}{\sigma \sqrt{2\pi t}} e^{-\frac{x - x_0 - \mu t}{2\sigma^2 t}}
\]
which is the density for a normal distribution with mean \( x_0 + \mu t \) and variance \( \sigma^2 t \).

In general when \( t_0 \) is not necessarily 0 we get
\[
\phi(x; t; x_0, t_0) = \frac{1}{\sigma \sqrt{2\pi (t - t_0)}} e^{-\frac{x - x_0 - \mu (t - t_0)}{2\sigma^2 (t - t_0)}}
\]