EIGENVECTOR NORMS MATTER IN SPECTRAL GRAPH THEORY

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Connections in Discrete Mathematics, A Celebration of Ron Graham

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WHAT IS SPECTRAL GRAPH THEORY?

Answer: Application of Linear Algebra, in particular eigenvalues and eigenvectors, to the study of graphs and networks.

Eigenvalues: \( \left\{ \frac{1}{2} (1 + \sqrt{17}) , \frac{1}{2} (1 - \sqrt{17}) , -1 , 0 \right\} \)

These aren't the juggling patterns you're looking for.
WHAT ARE EIGENVALUES GOOD FOR?

Community Detection

- Cheeger’s Inequality
- Principle Component Analysis
- Bipartitioning
- Dense Subgraphs

Graph Coloring

- Mixing Times / Expander Mixing Lemma
TYPES OF MATRICES

Adjacency Matrix

Eigenvalues: $\rho_{max} \leq \rho_2 \leq \ldots \leq \rho_{min}$

Normalized Laplacian Matrix

Eigenvalues: $0 = \lambda_0 \leq \lambda_1 \leq \ldots \lambda_{n-1}$
BASIC FACTS OF THE LAPLACIAN

\[ L = I - D^{-1/2} A D^{-1/2} \]

- \( \lambda_0 = 0 \) with eigenvector \( D^{1/2} 1 \)
- \( \lambda_{n-1} \leq 2 \) with equality if \( G \) has a bipartite component.
- \( \lambda_1 = 0 \) if and only if \( G \) is disconnected.
- \( \lambda_{k-1} = 0 \) if and only if \( G \) has at least \( k \) connected components.

\( \lambda_1 = 0.110307 \)
**CHEEGER CONSTANT**

\[ h(S) := \frac{E(S, \bar{S})}{\min(\text{vol } S, \text{vol } \bar{S})} \]

“Boundary”

“Area”

\[ h_G = \min_{S \subset V(G)} h(S) \] NP-Hard to compute
CHEEGER’S INEQUALITY

Theorem (Alon-Milman 1985)

\[ \frac{h_G^2}{2} \leq \lambda_1 \leq 2h_G \]
DRAWBACK TO CHEEGER’S INEQUALITY

Proposition

Under mild conditions, \( h_G \leq \frac{1}{2} (1 + o(1)) \)

Just flip a coin for each vertex…
Theorem (K. 2015+)

Let \( \lambda_1 \) be the second minimum eigenvalue of the normalized Laplacian with unit eigenvector \( \mathbf{D}^{1/2} \mathbf{v} \). Then, under mild conditions,

\[
\frac{1}{2} - \frac{1 - \lambda_1}{2} \leq h_G \leq \left( \frac{1}{2} - \frac{1 - \lambda_1}{2\| \mathbf{v} \|_{\infty}^2 \text{vol } G} \right) (1 + o(1)).
\]

_Cheeger constant_ is a measure of how much better than random you can get!
PROOF IDEA:

Just flip a **weighted** coin for each vertex...

...where the weight for vertex $i$ is given by

$$\left( \frac{1}{2} + \frac{v_i}{2\|v\|_\infty} \right) (1 - \varepsilon)$$
Lemma (Random Quadratic Forms)

Given a real symmetric matrix $A$. For a random vector $x$,

$$\mathbb{E}[x^*Ax] = \mu^*A\mu + \text{Tr}(\Sigma A)$$

If $A$ has diagonal zero entries, and $x$ has independent components...
Let $\lambda_1$ be the second minimum eigenvalue of the normalized Laplacian with unit eigenvector $D^{1/2}v$. Then, under mild conditions,

$$\frac{1}{2} - \frac{1 - \lambda_1}{2} \leq h_G \leq \left(\frac{1}{2} - \frac{1 - \lambda_1}{2\|v\|_\infty^2 vol G} \right) (1 + o(1)).$$
Corollary

If $\lambda_i = 1 + o(1)$ for $i \neq 0$ (i.e., $G$ is an expander), then under mild conditions:

$$h_G = \frac{1}{2} (1 + o(1))$$
Cheeger Constant is for 2 parts.

What about minimizing edges between 3, 4, or $k$ parts?

**Higher Order Cheeger Constant (Worst Case Approach):**

$$\hat{h}_G^{(k)} = \min_{\mathcal{P}=(S_1,S_2,\ldots,S_k)} \max_i h(S_i)$$
K-FOLD CHEEGER INEQUALITY

Theorem (Lee-Gharan-Trevisan 2012)

\[ \frac{\lambda_k}{2} \leq \hat{h}_G^{(k+1)} \leq O(k^2) \sqrt{2\lambda_k} \]

Similar work by Louis-Raghavendra-Tetali-Vempala (2011)

Can this be improved to a linear factor as previously?
K-FOLD CHEEGER CONSTANT: ANOTHER APPROACH

Higher Order Cheeger Constant (Average Case Approach):

\[ h_G^{(k)}(S) = \frac{1}{k} \sum_{i \neq j} \frac{e(S_i, S_j)}{\min\{\text{vol } S_i, \text{vol } S_j\}}. \]

Average over all parts over Cheeger ratio among those parts.

Low Average Case Cheeger Constant
High Worst Case Cheeger Constant
Theorem (K.-Radcliffe 2015+)

Let $\alpha = \sum_{i=1}^{k-1} \|x_i\|_\infty$, and let $\Lambda = \frac{1}{k} \sum_{i=1}^{k-1} (1 - \lambda_i)$

Then, under mild conditions,

$$\frac{1}{2} - \frac{\Lambda}{2} \leq h^{(k)}_G \leq \left[ \frac{1}{2} - \frac{1}{4k} - \frac{(k-1)\Lambda}{4\text{vol}(G) \alpha^2} \right] (1 + o(1)).$$

Also determines how much better than random you can get.

Uses similar, but more detailed, approach as before.
WHAT ELSE CAN YOU DO WITH EIGENVECTOR NORMS?

Densest Subgraph Problem

Expander Mixing Lemma

Graph Coloring
DENSE SUBGRAPH PROBLEM

Densest Subgraph Problem

\[ M := \max_{S \subseteq V, S \neq \emptyset} \frac{|E(S, S)|}{|S|} \]

Goldburg (1984): polynomial time algorithm to find the densest subgraph.

NP-complete when \( S \) has additional restrictions
DENSE SUBGRAPH PROBLEM

Densest Subgraph Problem

\[ M := \max_{S \subset V, S \neq \emptyset} \frac{|E(S, S)|}{|S|} \]

Theorem (K. 2015+)

\[ \frac{\rho_{\text{max}}}{\|v\|_1 \|v\|_\infty} \leq M \leq \rho_{\text{max}} \]

Corresponding Unit Eigenvector

Maximum Eigenvalue of \( A \)

Maximum Density
DENSE SUBGRAPH PROBLEM

Theorem (K. 2015+)

\[ M := \max_{S \subseteq V, S \neq \emptyset} \frac{|E(S, S)|}{|S|} \]

Densest Subgraph Problem

\[ \|v\|_1 \|v\|_\infty \geq \frac{\rho_{\text{max}} M}{M} \geq \|v\|_2^2 \]
Theorem (Bilu-Linial 2004)

Let $G$ be a $d$-regular undirected graph. Define $\beta$ is the least constant such that for all sets of vertices $S, T$:

$$|E(S, T) - \frac{d|S||T|}{n}| \leq \beta \sqrt{|S||T|}$$

then $\sigma_2 := \max[\rho_2, -\rho_{\min}]$ obeys

$$\beta \leq \sigma_2 \leq O\left(\beta + \beta \log \frac{d}{\beta}\right)$$

Theorem (K. ‘15+)

Let $G$ be a $d$-regular undirected graph. Define $\beta$ the be the least constant such that for all sets of vertices $S, T$:

$$\left| E(S, T) - \frac{d|S||T|}{n} \right| \leq \beta \sqrt{|S||T|}$$

then $\sigma_2 := \max[\rho_2, \rho_{\min}]$ with unit eigenvector $v$ obeys

$$\frac{\sigma_2}{\|v\|_1\|v\|_{\infty}} \leq \beta \leq \sigma_2$$

No logarithmic factor!
MAXCUT / ALMOST 2-COLORINGS

A “converse” applies. If the minimum eigenvalue is large, then a large subgraph has a large maximum cut.

\[ B = \max_{S_1 \cup S_2 = V} \frac{2E(S_1, S_2) - E(S_1, S_1) - E(S_2, S_2)}{n} \]

\[ B \leq -\rho_{\text{min}} \]

Trevisan (2008), Bauer-Hua-Jöst (2012):
MAXCUT / ALMOST 2-COLORINGS

Maximum Cut Problem

\[ B = \max_{S_1 \cup S_2 = V} \frac{2E(S_1, S_2) - E(S_1, S_1) - E(S_2, S_2)}{n} \]

Theorem (K. 2015+)

\[ \frac{-\rho_{\min}}{n \|v\|_\infty} \leq B \leq -\rho_{\min} \]

Corresponding Unit Eigenvector

Minimum Eigenvalue of A
MORAL

- Many spectral bounds are tight when corresponding eigenvectors are “well-behaved.”
- It’s okay not to be “norm”al.