PSET 3 Solns

Paul Hand

18.085

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1) Gram Schmidt on $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

will give $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\-1\end{pmatrix}$ (before normalization)

After normalization: $\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}$

We write $A = \begin{pmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\1 & 0 & 0\\0 & 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}r_{11} & r_{12} & r_{13}\\ & r_{21} & r_{23}\\ & & r_{33}\end{pmatrix}$

Matching left column: $r_{11}=1$

Matching 2nd column $r_{12}\begin{pmatrix}0\\1\\0\end{pmatrix}+r_{21}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix} \Rightarrow r_{12}=1$, $r_{21}=\sqrt{2}$

Matching 3rd column $r_{13}\begin{pmatrix}0\\1\\0\end{pmatrix}+r_{23}\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}+r_{33}\begin{pmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix} \Rightarrow$ $r_{13}=1$, $r_{23}=0$, $r_{33}=\sqrt{2}$

$$A = \begin{pmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\1 & 0 & 0\\0 & 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 1 & 1\\ & \sqrt{2} & 0\\ & & \sqrt{2}\end{pmatrix}$$

Solving $Ax=\begin{pmatrix}4\\5\\0\end{pmatrix}$ $\quad QRx=\begin{pmatrix}4\\5\\0\end{pmatrix}$

$$Rx = Q^t\begin{pmatrix}4\\5\\0\end{pmatrix} = \begin{pmatrix}5\\2\sqrt{2}\\2\sqrt{2}\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 1\\ & \sqrt{2} & 0\\ & & \sqrt{2}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}5\\2\sqrt{2}\\2\sqrt{2}\end{pmatrix}$$

$x_3 = 2$

$x_2 = 2$

$x_1+x_2+x_3=5 \Rightarrow x_1 = 1$

$$\boxed{x=\begin{pmatrix}1\\2\\2\end{pmatrix}}$$

2 a) We show both Eigenvalues positive if $a > 0$ & $ac - b^2 > 0$.

Observe $c > 0$, as otherwise $ac - b^2 < 0$.

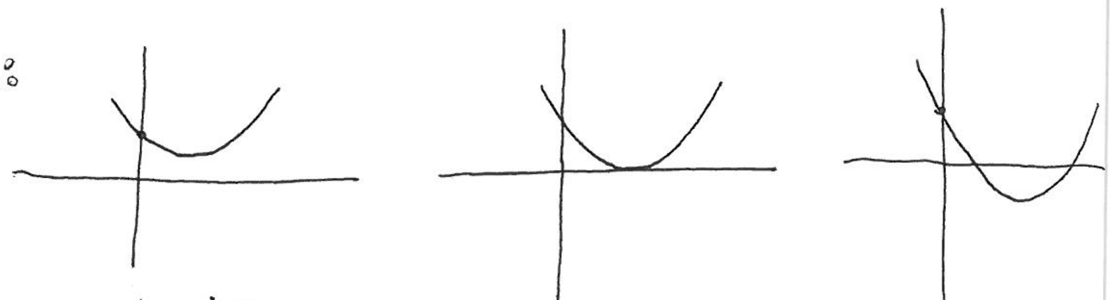
Eigenvalues are given by roots of

$$(\lambda - a)(\lambda - c) - b^2 = 0$$

$$\lambda^2 - (a+c)\lambda + ac - b^2 = 0 \qquad (*)$$

This is a parabola – with minimum value at $\lambda = \frac{a+c}{2}$.
– with positive value $ac - b^2$ at $\lambda = 0$

Three cases:



min value > 0 | min value $= 0$ | min value < 0.

In this case, roots are complex, not pos def.

In these cases roots are positive and matrix is pos def.

Suffices to show min value ≤ 0:

Plug in $\lambda = \frac{a+c}{2}$ to $\lambda^2 - (a+c)\lambda + ac - b^2 = f(\lambda)$

$$f\left(\frac{a+c}{2}\right) = \left(\frac{a+c}{2}\right)^2 - \frac{(a+c)^2}{2} + ac - b^2$$

$$= -\frac{(a+c)^2}{4} + ac - b^2 = -\frac{(a-c)^2}{4} - b^2 \leq 0.$$

2b) A is sym, pos def

$x^t A x \geq 0$ for all x.

Let $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$x^t A x = A_{11} \geq 0.$

3a) $A^tA\,\hat{u} = A^tb$

$A^tA = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad A^tb = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Normal Eqns $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \hat{u} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$$\hat{u} = \begin{pmatrix} 4/3 \\ 3/2 \end{pmatrix}$$

b) Gram Schmidt of $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$ gives $\begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$

(two cols are already perpendicular)

Observe $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

$$A = \hat{Q} \quad \tilde{R}$$

$\hat{R}\,\hat{u} = \bar{Q}^t b$

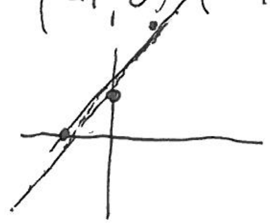
$\begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \hat{u} = \begin{pmatrix} 4/\sqrt{3} \\ 3/\sqrt{2} \end{pmatrix}$

$$\hat{u} = \begin{matrix} 4/3 \\ 3/2 \end{matrix}$$

c) $\text{cond}(A^tA) = \dfrac{\lambda_{max}}{\lambda_{min}} = 3/2$ ← worse

$\text{cond}(\hat{R}) = \dfrac{\lambda_{max}}{\lambda_{min}} = \sqrt{3}/\sqrt{2}$

d) Find best fit line through $(-1,0)$, $(0,1)$ $(1,3)$



4)

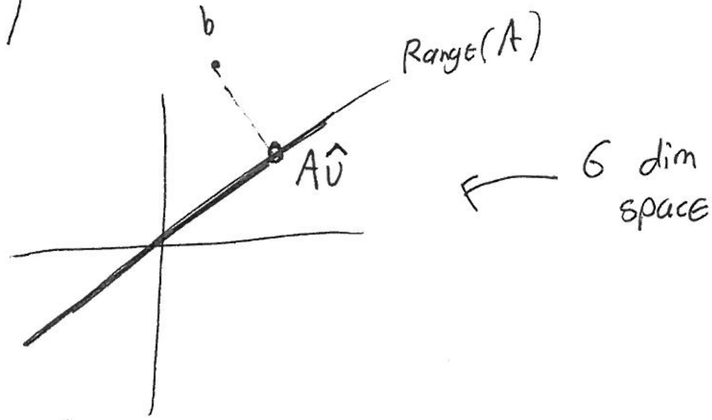
~~[illegible]~~ point given by finding $\min_{u} \|Au - b\|^2$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$

$\hat{u} = A \backslash b = \begin{pmatrix} 3.5 \\ -1.5 \\ 0 \end{pmatrix}$

These are coefs in basis of 3 vectors.

$A\hat{u} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 5 \\ 5 \\ 5 \end{pmatrix}$ is nearest point



5) a) yes. $(XX^t)^t = (X^t)^t X^t = XX^t$

b) Range of A is line spanned by X.

Observe X is eigenvector of eigenvalue $|X|^2 = 55$

$$AX = XX^tX = X(X^tX) = X|X|^2 = 55X.$$

The rank of A is $\underline{1}$. Dim of null space $= 4$.

Any vector in null space is eigen vector of eigenvalue 0.

Hence we need only find basis of null space.

Null space is set of all vectors perpendicular to rows of A. All rows are multiples of X^t.

Hence we find all vectors perp to $\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \cdot \begin{pmatrix}v\\w\\x\\y\\z\end{pmatrix} = 0 \implies v + 2w + 3x + 4y + 5z = 0.$$

5 vars, $\underline{1}$ constraint.

All solns of form

$$\begin{pmatrix}-2w-3x-4y-5z\\w\\x\\y\\z\end{pmatrix} = w\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + x\begin{pmatrix}-3\\0\\1\\0\\0\end{pmatrix} + y\begin{pmatrix}-4\\0\\0\\1\\0\end{pmatrix} + z\begin{pmatrix}-5\\0\\0\\0\\1\end{pmatrix}$$

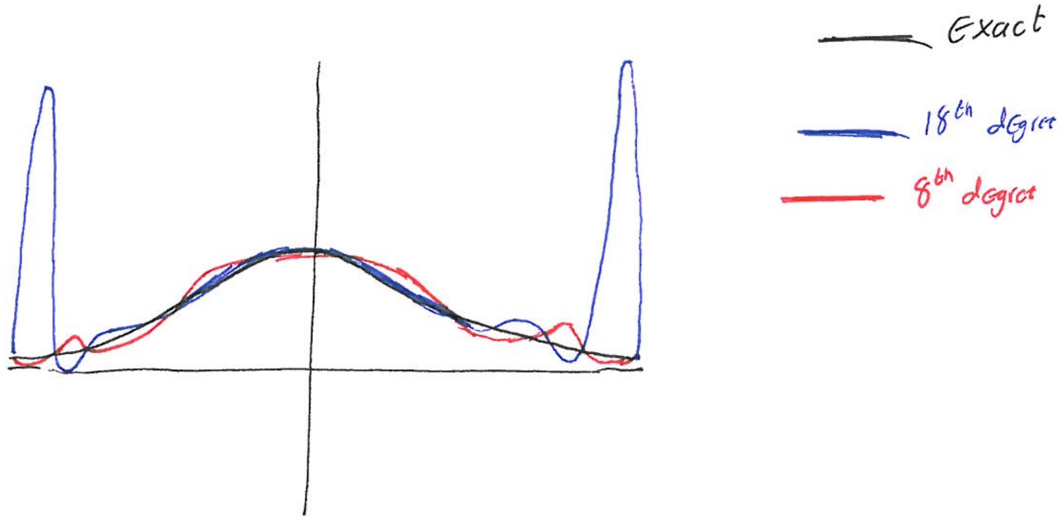
Eigenbasis is:

$\lambda = 55$: $\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}$ $\qquad \lambda = 0$: $\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}-3\\0\\1\\0\\0\end{pmatrix}\begin{pmatrix}-4\\0\\0\\1\\0\end{pmatrix}\begin{pmatrix}-5\\0\\0\\0\\1\end{pmatrix}$

6 a) See code $N = 20 \quad M = 18$

b) $N = 20 \quad M = 8$

c)



d) The 18th degree polynomial has less residual (because all 8th degree polynomials are also 18th degree polynomials with zero's for large powers.)

The 18th degree polynomial is great at approximating the data in middle of domain

The 8th degree polynomial is better at approximating the data near the boundary.