PSET 4 Solutions

1

a) $X = \text{linspace}(-1, 1, 50)$

$C_{\text{computed}} = \text{vander}(X) \backslash X$

$C_{\text{actual}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} = \text{zeros}(50,1);\ C_{\text{actual}}(49) = 1$

$$\text{fractional error} = \frac{\text{norm}(C_{\text{computed}} - C_{\text{actual}})}{\text{norm}(C_{\text{actual}})}$$

On my machine, this is 0

b) $C_{\text{computed}} = \text{vander}(X) \backslash (\frac{1}{3} * X)$

$C_{\text{actual}} = \text{zeros}(50,1) \quad C_{\text{actual}}(49) = 1/3$

fractional error $= 1.96 \cdot 10^{5}$ (thats a large fraction!)

c) $\text{Cond}(\text{vander}(X)) = 2.4 \cdot 10^{18}$

Rounding error on order of 10^{-16}

This condition number would lead us to expect fractional error of coefs on order of 10^{2}.

There must be additional sources of error

2

a) The rows add up to 1 means

$$A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

Hence $\begin{pmatrix}1\\1\end{pmatrix}$ is an eigenvector of eigenvalue 1

Because A is symmetric, it has an orthonormal basis of eigenvectors.

Thus $\begin{pmatrix}-1\\1\end{pmatrix}$ must be an eigenvector

$$A\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-1/2\\1/2\end{pmatrix}$$

Hence $\begin{pmatrix}-1\\1\end{pmatrix}$ is eigenvector of eigenvalue $1/2$.

Normalizing and putting in form of eigenvalue decomposition

$$A=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$$

b) $A = Q\Lambda Q^*$

$$A^n = \underbrace{Q\Lambda Q^*\, Q\Lambda Q^* \cdots Q\Lambda Q^*}_{n\text{ times}}$$

Because Q is orthogonal, $Q^*Q=I$

$$A^n = Q\Lambda^n Q^*$$

Note $\Lambda^n=\begin{pmatrix}1^n & 0\\ 0 & (\frac{1}{2})^n\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & \frac{1}{2^n}\end{pmatrix}$

As $n\to\infty$, $A^n \to \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}=\begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}$

3) Rank of $A = 1$

Thus reduced SVD of A is

$$A = \begin{pmatrix} | \\ U_1 \\ | \end{pmatrix} (\sigma_1) (-v_1^*-) \qquad \text{where } U_1 \text{ is } 3\times 1, \quad V_1 \text{ is } 2\times 1$$

The range of A is the span of U_1.

Observing from $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, the range of A is span of $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$.

Normalizing gives $U_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The find the right singular vector, we first find the null space of A, which is any multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

The first right singuler vector must be perpendiculr to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and normal, thus, it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

Observing $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ we see $\sigma_1 = 4$

Thus $A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (4) (0 \ 1)$ is reduced SVD.

To make this a full SVD, we add the remaining orthonormal colums to U and V

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b) The null space is spanned by the right singuler vectors of singular value zero.

$$A v_i = \sigma_i U_i \implies V_i \text{ in } Null(A) \text{ if } \sigma_i = 0$$

These right singuler vectors are $\begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$,

4) $V_j(k) = e^{2\pi i j k/N}$

To show V_j is an Eigenvector,

show $(AV_j)(k) = \lambda_j V_j(k)$

$$AV_j(k) = \sum_m A_{km} V_j(m)$$

$$= -1\, V_j(k-1) + 2\, V_j(k) - 1\, V_j(k+1)$$

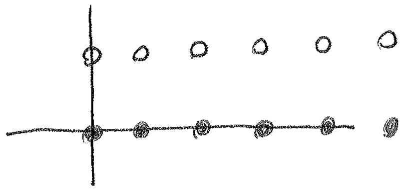
$$= -e^{2\pi i j(k-1)/N} + 2e^{2\pi i j k/N} - e^{2\pi i j(k+1)/N}$$

$$= e^{2\pi i j k/N}\left[-1\, e^{-2\pi i j/N} + 2 - e^{2\pi i j/N}\right]$$

$$= e^{2\pi i j k/N}\left[2 - 2\cos 2\pi j/N\right]$$

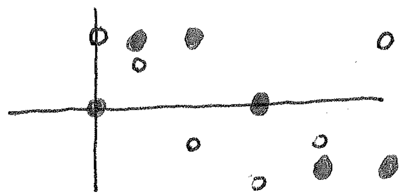
Hence V_j is Eigenvector of Eigen value $2 - 2\cos 2\pi j/N$

5a

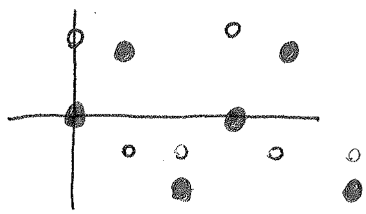


j=0

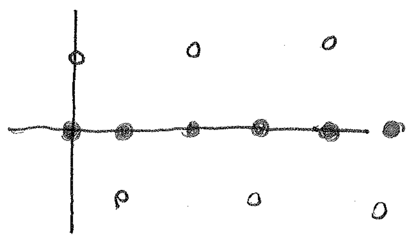
— real part ○
— imaginary part ●



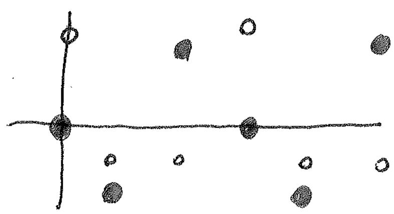
j=1



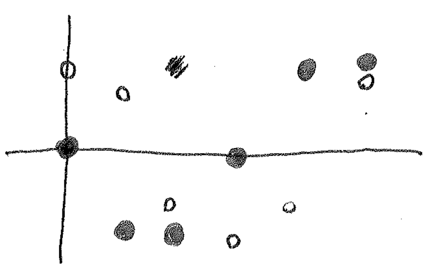
j=2



j=3

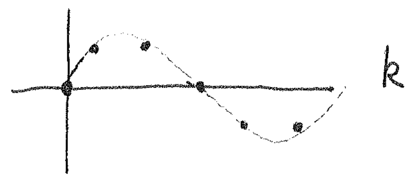


j=4



j=5

5b) $X = \begin{pmatrix}0\\1\\1\\0\\-1\\-1\end{pmatrix}$



$$X(k) = \frac{\sin 2\pi \frac{k\cdot 1}{6}}{\sin 2\pi/6} = \frac{2}{\sqrt{3}} \sin 2\pi \, k/6$$

$$= \frac{2}{\sqrt{3}} \left[\frac{e^{2\pi i \, k/6}}{2i} - \frac{e^{-2\pi i \, k/6}}{2i} \right]$$

$F_N\left(\frac{1}{N}\hat{X}\right) = X.$

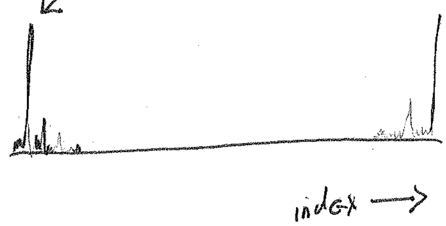
So $\frac{1}{6}\hat{X} = \begin{pmatrix}0\\ \frac{1}{\sqrt{3}i}\\0\\0\\0\\ \frac{-1}{\sqrt{3}i}\end{pmatrix}$ $\qquad \hat{X} = \begin{pmatrix}0\\-2\sqrt{3}i\\0\\0\\0\\2\sqrt{3}i\end{pmatrix}$

6) fft of signal looks like:

$X =$ wavread('single_note.wav')

plot(abs(fft(X)))

at (1-indexed) position 294.



This corresponds to wave that oscillates

293 times in 66150 index positions.

The sampling rate is 44100 $\frac{\text{sample}}{\text{sec}}$,

So the wave oscillates at

293 times in 1.5 sec

or 195.3 Hz.

A piano table gives this as G_3

(though I actually hit a G of neighboring octave)