PSET 5

1

a) $\int_{-\infty}^{x} \delta(y-2)\, dy$

If $x<2$, region of integration doesnt include singularity,
$\int_{-\infty}^{x} \delta(y-2)\, dy = 0$

If $x>2$, $\int_{-\infty}^{x} \delta(y-2) = 1.$

So $\int_{-\infty}^{x} \delta(y-2)\, dy = \mathcal{H}(x-2)$

b) $\int_{-\infty}^{x} \delta(y-3)\, dy = \mathcal{H}(x-3)$

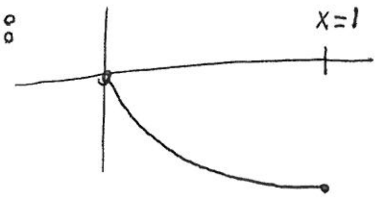
$\int_{-\infty}^{x} \delta(y+2)\, dy = \mathcal{H}(y+2)$

So $\int_{-\infty}^{x} \delta(y-3) - \frac{1}{2}\delta(y+2)\, dy = \mathcal{H}(x-3) - \frac{1}{2}\mathcal{H}(y+2)$

2

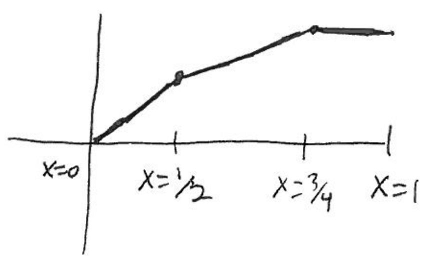
a) Constant force down : $f = -1$

$U(x)$:

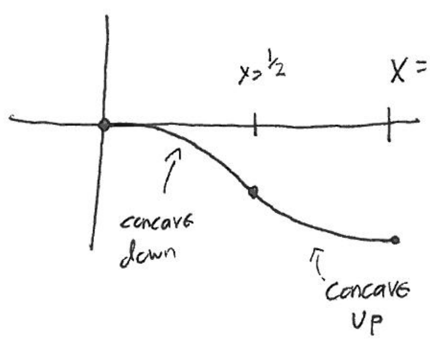


b) Slope is 0 on right, constant ~~between~~ except at $x = \frac{1}{2}, \frac{3}{4}$

Force is up at $x = \frac{1}{2}, \frac{3}{4}$



c) Force is up for $x < \frac{1}{2}$
down for $x > \frac{1}{2}$



How do we know $u'(0) = 0$?

Integrate ODE

$$\int_0^{L/2} -\frac{d^2 u}{dx^2} = \int_0^{L/2} f$$

$$\frac{du}{dx}(0) - \frac{du}{dx}(L/2) = L/2$$

$$\int_{L/2}^{L} -\frac{d^2 u}{dx^2} = \int_{L/2}^{L} f$$

$$\frac{du}{dx}\left(\frac{L}{2}\right) - \frac{du}{dx}(L) = -L/2$$

$$\Rightarrow \frac{du}{dx}(0) - \frac{du}{dx}(L) = 0.$$

3)

$$\begin{cases} -\dfrac{d^2 U}{dx^2} = \sin\dfrac{\pi x}{L} \\ U(0) = 0 \\ U(L) = 0 \end{cases}$$

Integrating

$$U(x) = \frac{L^2}{\pi^2} \sin\frac{\pi x}{L} + Cx + d$$

$U(0) = 0 \Rightarrow d = 0$

$U(L) = 0 \Rightarrow C = 0$

$$U(x) = \frac{L^2}{\pi^2} \sin\frac{\pi x}{L}$$

4)

$$\begin{cases} -\dfrac{d^2U}{dx^2} = \delta(x - L/2) \\ U(0) = 0 \\ U'(L) = 0 \end{cases}$$

$$U(x) = \begin{cases} ax + b & \text{if } x < L/2 \\ cx + d & \text{if } x > L/2 \end{cases}$$

as $\dfrac{d^2U}{dx^2} = 0$ away from $x = L/2$

$U(0) = 0 \Rightarrow b = 0$

$U'(L) = 0 \Rightarrow c = 0$

$$U(x) = \begin{cases} ax & \text{if } x < L/2 \\ d & \text{if } x > L/2 \end{cases}$$

Apply continuity:

$$a\frac{L}{2} = d$$

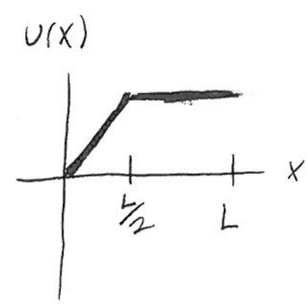
Jump condition

$$-\left[\frac{dU}{dx}\right]_{L/2} = 1$$

$$-(0 - a) = 1$$

$$a = 1$$

$$d = L/2$$



$$U(x) = \begin{cases} x & x \leq L/2 \\ L/2 & x \geq L/2 \end{cases}$$

5)

a i) $$\frac{y^{n+1}-y^n}{\Delta t} = Ay^n$$

$$y^{n+1} = y^n + A\,\Delta t\, y^n$$

$$y^{n+1} = (I + A\,\Delta t)\,y^n$$

ii) $$\frac{y^{n+1}-y^n}{\Delta t} = Ay^{n+1}$$

$$y^{n+1} - \Delta t A y^{n+1} = y^n$$

$$(I - \Delta t A)\,y^{n+1} = y^n$$

$$y^{n+1} = (I - \Delta t\, A) \backslash y^n$$

iii) $$\frac{y^{n+1}-y^n}{\Delta t} = \frac{Ay^{n+1} + Ay^n}{2}$$

$$y^{n+1} - \tfrac{\Delta t}{2} A y^{n+1} = y^n + \tfrac{\Delta t}{2} A y^n$$

$$y^{n+1} = (I - \tfrac{\Delta t}{2} A) \backslash (I + \tfrac{\Delta t}{2} A)\,y^n$$