

## Day 18 — Summary — Series

1. If  $\{x_n\}$  is a sequence in a normed vector space, we define the infinite sum  $\sum_{n=1}^{\infty} x_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n$ . The infinite series converges if this sum exists. We say that an infinite series diverges if the partial sums are unbounded.
2. Comparison test. Let  $\sum a_n$  and  $\sum b_n$  be series of real numbers. If  $\sum b_n$  converges and  $0 \leq a_n \leq b_n$  for sufficiently large  $n$ , then  $\sum a_n$  converges.
3. Ratio test. Let  $\sum a_n$  be a series of nonnegative real numbers, and let  $0 < c < 1$  be such that  $a_{n+1} \leq ca_n$  for sufficiently large  $n$ . Then  $\sum a_n$  converges.
4. Integral test. Let  $f$  be a decreasing function over all real numbers  $\geq 1$ . The infinite series  $\sum_{n=1}^{\infty} f(n)$  converges if and only if  $\int_a^{\infty} f(x)dx$  exists and is finite. Note that  $\int_a^{\infty} f(x)dx$  is defined as  $\lim_{M \rightarrow \infty} \int_1^M f(x)dx$ .
5. Let  $\sum a_n$  be a series of numbers. If  $\sum |a_n|$  converges, then  $\sum a_n$  converges. The series  $\sum a_n$  is said to converge absolutely if  $\sum |a_n|$  converges.
6. Let  $\{a_n\}$  be a sequence of numbers monotonically decreasing to zero. The alternating series  $\sum (-1)^n a_n$  converges.

Warmup:

Provide a cont function  $f$  over  $\mathbb{R}^{\infty}$   
a closed S. such that  $\max f$  is  
finite & not achieved

1) Converges:

$$\sum_{n=1}^{\infty} a^n \quad \text{for } |a| < 1 \quad (\text{direct computation})$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{for } p > 1 \quad (\text{integral test})$$

ratio  
integral

Diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{for } p \leq 1$$

$$\sum_{n=1}^{\infty} a^n \quad \text{for } \cancel{|a| \neq 1} \quad a \in (-\infty, 1) \cup [1, \infty)$$

Example: Let  $S_N = \sum_{i=0}^N a^n$

$$S_N = 1 + a + \dots + a^N$$

$$aS_N = \quad a + \dots + a^N + a^{N+1}$$

$$(a-1)S_N = a^{N+1} - 1$$

$$S_N = \frac{a^{N+1} - 1}{a - 1}$$

$$\lim_{N \rightarrow \infty} S_N = \frac{1}{1-a} \quad \text{if } |a| < 1$$

2) Let  $a_n, b_n \rightarrow 0$  for  $a_n \leq b_n$  for  $n \geq N$ .

Then  $\sum_{n=1}^{\infty} b_n$  converges  $\Rightarrow \sum_{n=1}^{\infty} a_n$  converges

$\sum_{n=1}^{\infty} a_n$  diverges  $\Rightarrow \sum_{n=1}^{\infty} b_n$  diverges

Example:

$\sum_{n=1}^{\infty} \frac{\log n}{n}$  diverges by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n}$ .

Example:  $\sum_{n=1}^{\infty} \frac{\log^3 n}{n^2}$  converges, by comparison to  $\sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$   
Because  $\lim_{n \rightarrow \infty} \frac{\log^3 n}{\sqrt{n}} = 0$ ,  $\exists N$  st  $\log^3 n \leq \sqrt{n}$

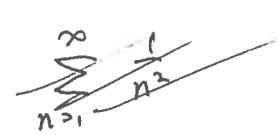
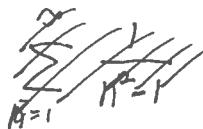
Example:  $\sum_{n=1}^{\infty} n^3 e^{-n}$  converges by comp to  $\sum_{n=1}^{\infty} e^{-n/2}$

3) If  $0 \leq a_{n+1} \leq c a_n$  for  $c \in (0, 1)$  and  $n \geq N$ ,

then  $\sum_{n=1}^{\infty} a_n$  converges.

Pf: Compare to  $\sum_{n=1}^{\infty} c^n$  (or  $a_N \sum_{n=1}^{\infty} c^{(n-N)}$ )

Example:



$$\text{In Note } a_n = \frac{1}{n!}$$

$$\frac{a_{n+1}}{a_n} =$$

$$\sum_{n=1}^{\infty} \frac{b^n}{n!} \text{ converges} \quad a_n = \frac{b^n}{n!} \quad \frac{a_{n+1}}{a_n} = \frac{b^{n+1}}{(n+1)!} \quad \frac{n!}{b^n} = \frac{b_n}{n!} \rightarrow 0$$

Ratio test applies for  $c = \frac{1}{2}$ , e.g.

5) Let  $a_n \in \mathbb{R}$   $\sum_{n=1}^{\infty} |a_n|$  converges  $\Rightarrow \sum_{n=1}^{\infty} a_n$  converges

Iden<sup>o</sup> Worst case of sum is when everything adds  
(no cancellation)

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof<sup>o</sup> Let  $S_N = \sum_{n=1}^N a_n$  Let  $T_N = \sum_{n=1}^N |a_n|$   
 $|S_N| = \left| \sum_{n=1}^N a_n \right| \leq \sum_{n=1}^N |a_n| = T_N$  bound.

So  $S_N$  is bdd and  $\uparrow$ . So  $S_N$  conv.

Ex  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$  conv bc  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  conv