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Analysis I
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Day 3 — Summary — Limits and continuity of functions

15. Let f be a function defined on $S \subset \mathbb{R}$. The limit of $f(x)$ as x approaches a exists if there exists an L such that for all ε there is a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ for $x \in S$. We write such a limit as $\lim_{x \to a} f(x) = L$.

16. Limits commute with addition, multiplication, division, and non-strict inequalities

    (a) If $\lim_{x\to a}(cf)(x) = c \lim_{x\to a} f(x)$ for any real c.

    (b) If $\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ if both limits on the right exist.

    (c) If $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ if both limits on the right exist.

    (d) If $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/\lim_{x\to a} g(x)$ if both limits on the right exist and the limit of g is nonzero.

    (e) If $f(x) \leq g(x)$ for all x sufficiently close to a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided both limits on the right exist.

17. The function $f : S \to \mathbb{R}$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

18. The function f is continuous on the set S if f is continuous at every point in S.

19. The composition of two continuous functions is continuous.

20. Intermediate value theorem: Let f be continuous on $[a, b]$. For any y satisfying $f(a) < y < f(b)$ or $f(b) < y < f(a)$, there exists an $x \in (a, b)$ such that $f(x) = y$.

21. The function f is uniformly continuous on the set S if for all ε, there exists a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Notice that the dependence of δ on ϵ does not depend on the position within the set. That is what makes it uniform.

22. A continuous function on a closed, bounded interval is uniformly continuous.

Warmup:

Example of sequences x_n, y_n satisfying

$x_n \to 0$

$y_n \to 0$

$\frac{x_n}{y_n} \to a$

Example:

$x_n \to 0$

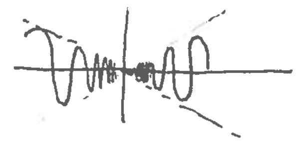
$y_n \to \infty$

$x_n y_n$ has no limit yet is still bounded

15) Let $f: S \to \mathbb{R}$. $\lim_{x \to a} f(x) = L$ means $\forall \varepsilon \, \exists \delta$ st $|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$

Conceptually: f ~~gets~~ is arbiturily close to L for value of x sufficient, near a.

Example: Let $f(x) = \begin{cases} 0 & \text{if } x=0 \\ x\sin\frac{1}{x} & \text{if } x\neq 0 \end{cases}$



Claim: $\lim_{x\to 0} f(x) = 0$.

Fix ε. Let $\delta = \varepsilon$. If $|x|<\delta$, then

$$|f(x)-L| = \begin{cases} |x\sin\frac{1}{x} - 0| & \text{if } x\neq 0 \\ |0-0| & \text{if } x=0 \end{cases}$$

$$\leq \begin{cases} |x|\,|\sin\frac{1}{x}| & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$\leq |x| < \delta = \varepsilon \blacksquare$$

16c) If $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ (finite) then $\lim_{x\to a} f(x)g(x) = LM$

Getting at the Proof: Know: $|f(x)-L|$ small when x near a
$|g(x)-M|$ small when x near a

Want: $|f(x)g(x)-LM|$ small when x near a

Consider $|f(x)g(x) - L g(x) + L g(x) - LM|$

$$= |\underbrace{(f(x)-L)}_{\text{small}}\underbrace{g(x)}_{\text{finite}} + \underbrace{L}_{\text{finite}}\underbrace{(g(x)-M)}_{\text{small}}| \leq \frac{\varepsilon}{2C} C + \frac{\varepsilon}{2C} = \varepsilon$$

Need to argue g is bdd.
Let $\varepsilon = 1$. $\exists\, \delta$ st $|x-a|<\delta \Rightarrow |g(x)-M| < 1$
$\Rightarrow |g(x)| \leq 1 + |M|$.

Proof: We need to show $\forall \varepsilon\ \exists \delta$ st $|x-a|<\delta \Rightarrow |fg - LM| < \varepsilon$.
By $\lim_{x\to a} g(x) = M$, $\exists\, \delta_0$ st $|g(x)| \leq 1+M$. for all $|x-x_0| < \delta_0$.
Choosing $C = \max(1+M, L)$; Let δ_f & δ_g be from dfn of $\lim_{x\to a} f = L$, $\lim_{x\to a} g = M$. for $\varepsilon/2C$
Fix ε. Let $\delta = \min(\delta_0, \delta_f, \delta_g)$

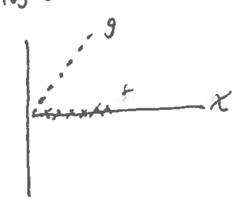
Then $|f(x)g(x) - LM| = |(f(x)-L)g(x) + L(g(x)-M)|$
$\leq |f(x)-L|\,|g(x)| + |L|\,|g(x)-M|$
$\leq \frac{\varepsilon}{2C} C + C\frac{\varepsilon}{2C} = \varepsilon$ ∎

16 e) $f(x) \leq g(x)$ for all $|x-a| < \delta \Rightarrow \lim_{x\to a} f(x) \leq \lim_{x \to a} g(x)$ if both limits exist.

~~Example:~~

~~Why cant we~~

Can we make a corresponding statement with strict inequalities?



$f(x) < g(x) \Rightarrow \lim f < \lim g$?? No

Let $S = (0, \infty)$

$f(x) = 0$, $g(x) = x$

On S, $f(x) < g(x)$. But $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$.

Why dont we say $f(x) \leq g(x)$ for all $x \Rightarrow \lim f(x) \leq \lim g(x)$ if both limits exist.

With limits as $x \to a$, nothing that is a finite distance away from $x=a$ matters. All that matters is that $f \leq g$ "near" $x=a$.

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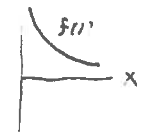
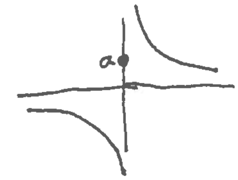
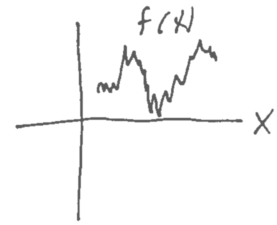
Examples: $f(x) = \frac{1}{x}$ is continuous on $(0, \varepsilon)$



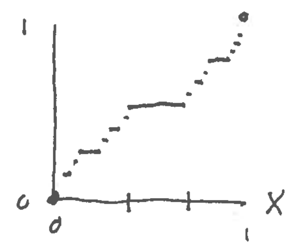
$$f(x) = \begin{cases} 1/x & x \neq 0 \\ a & x = 0 \end{cases}$$ is not continuous on $\mathbb{R}$ for any a.



Advanced example: Brownian motion
(Continuous random walk)
infinite arc length



Cantor function
continuous & monotonic increasing
increases despite be flat almost everywhere

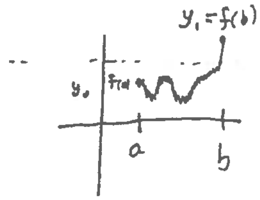


To specify $c(x)$:

1) write x in base 3
2) If x contains a 1, replace all digits subsequent to first 1 by 0
3) Replace all 2's with 1's
4) Interpret as binary #.

20) Intermediate Value Theorem

Conceptually: For $y=f(x)$
If you go from y_0 to y_1, you must pass every value in between.



... must cross this line somewhere

Application: Consider a track with continuous temperature. Is ~~there~~ necessarily a pair of opposing points with equal temperature?

Let f ~~(x)~~ $: \mathbb{R} \to \mathbb{R}$ be L periodic and continuous

$$f(x+L) = f(x).$$

Must there be an x such that $f(x+\frac{L}{2}) = f(x)$?

Yes. Consider $g(x) = f(x+\frac{L}{2}) - f(x)$, continuous.

~~$g(0)$~~

~~$f(\frac{L}{2}) = f(L$~~

Claim: $g(0) = -g(L/2)$

Note: $f(\frac{L}{2}) - f(0) = -\left(f(\frac{L}{2}+\frac{L}{2}) - f(\frac{L}{2})\right)$

Either $g(0) = -g(\frac{L}{2}) = 0$ or IVT guarantees $\exists\, x$ st $g(x)=0$ ∎

~~Proof of IVT (Sketch)~~

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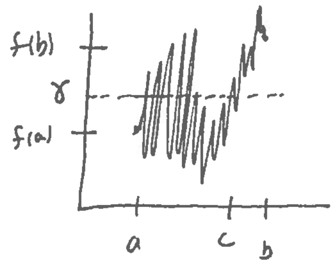
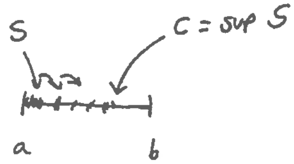


c is given by supremum of set of points w/ value $\leq \gamma$.



Let $S = \{ x \mid f(x) \leq \gamma \}$

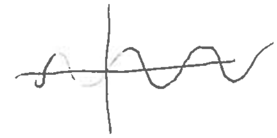
By continuity $\quad f(c) = \lim\limits_{\substack{x \to c \\ x \in S}} f(x) \leq \gamma$

For any $x > c$, $f(x) > \gamma$. So $\quad f(c) = \lim\limits_{\substack{x \to c \\ x \in T = (c, b]}} f(x) \geq \gamma$.

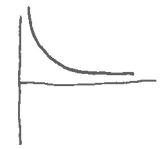
So $\quad f(c) = \gamma$ ∎

21,22 7/8 Uniform Continuity

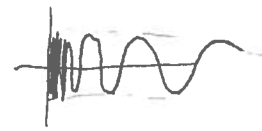
Examples: $f(x) = \sin x$ is unif cont on $\mathbb{R}$



$f(x) = \frac{1}{x}$ on $\{x > 0\}$ not uniformly continuous



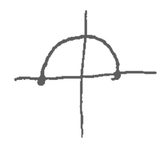
$f(x) = \sin(\frac{1}{x})$ on $\{x > 0\}$ not uniformly continuous



Is uniform continuity saying something about slope (if it exists)?

Sort of, but no.

$f(x) = \sqrt{1-x^2}$ is uniformly continuous on $[-1, 1]$
and has infinite slope near $x = \pm 1$.



Proof that $f \in C[a,b] \Rightarrow f$ is uniformly continuous on $[a,b]$ (sketch)

By contradiction & Bolzano Weierstrass

If not, $\exists\, x_n, y_n$ st $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| > \varepsilon$

By BZW, $x_{(n)} \to x^*$

By BZW $x_{((n))} \to x^*$ & $y_{((n))} \to y^*$. $y^* = x^*$ by $|x_n - y_n| \to 0$

By continuity $f(x_{((n))}) \to f(x^*) = f(y^*) \leftarrow f(y_{((n))})$.

contradicts $|f(x_n) - f(y_n)| > \varepsilon$ ■