

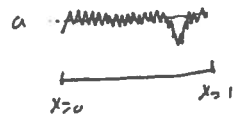
Day 5 — Summary — Differentiability, Mean Value Theorem

28. The derivative of f at x is $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, if this limit exists. A function is differentiable on a set if it is differentiable at every point in that set.
29. Product rule, quotient rule, chain rule.
30. Differentiability implies continuity.
31. Let $C^p([a, b])$ be the set of functions defined on $[a, b]$ that are differentiable p times, and the p -th derivative is continuous. Let C^∞ be the set of functions that are in C^p for all p .
32. At a local maximum (or minimum) of a differentiable function, the derivative is zero (provided that this max or min occurs in the interior of the function's domain).
33. Mean value theorem: If f is continuous on $[a, b]$ and is differentiable on (a, b) , then for some $c \in (a, b)$,
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Warm up

a) $f \in C(0,1)$

Can f cross value a only many times?



yes $f(x) = \sin \frac{1}{x}$ crosses 0 only many times

b) $f \in C[0,1]$

Can f cross value a only many times?

yes $f(x) = x \sin \frac{1}{x}$ crosses 0 only many times

c) $f \in C(0,1)$

Can f cross $[-\epsilon, \epsilon]$ an ∞ # times for a fixed $\epsilon > 0$?

yes $f(x) = \sin \frac{1}{x}$ crosses $(-\frac{1}{2}, \frac{1}{2})$ only many times

d) $f \in C[0,1]$

same question

no. Take points where $f(x) = a$. By Bolzano Weierstrass, a subseq converges to some $x^* \in [0,1]$. By continuity, $f(x^*) = a$, yet there are arbitrarily close x that do not have value a .

28)

Differentiability examples:

Continuous on $[0,1]$

Diffable on $(0,1)$ but not $[0,1]$:

$$\sqrt{1-x^2}, \quad \sqrt{x}$$

Defined on \mathbb{R} but diffable nowhere

$$\mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Defined on \mathbb{R} and diffable only at $x=0$

$$x^2 \mathbb{1}_{\mathbb{Q}}(x)$$

31) Examples

a) $f \in C^0(\mathbb{R})$ \circ
 $f \notin C^1(\mathbb{R})$ \circ

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$

b) $f \in C^p(\mathbb{R})$ \circ
 $f \notin C^{p+1}(\mathbb{R})$ \circ

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^p & x > 0 \end{cases}$$

c) $f \in C^\infty(\mathbb{R})$ \circ
 f is bounded \circ
 f is not constant \circ

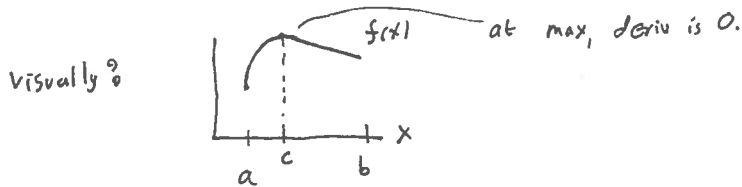
$$f(x) = \begin{matrix} \sin x \\ \text{or} \\ e^{-x^2} \end{matrix}$$

~~a) $f \in C^1(0,1)$ \circ
 $f \notin C^1[0,1]$ \circ~~

32)

If f has ^{local} max at X , ^{interior point} and f diff'able, $f'(x) = 0$.

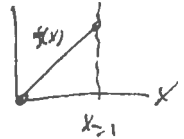
Formally: If f diff'able on (a, b) and $f(c) \geq f(x) \forall x \in (a, b)$
 then $f'(c) = 0$.



Applications: Find optimizers by setting derivative to 0.

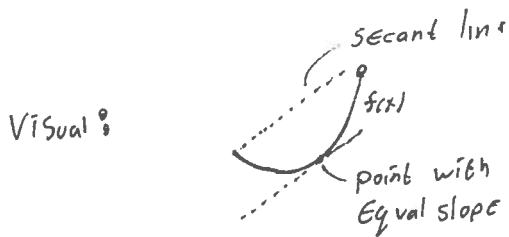
Proof: (sketch) $\frac{f(x+h) - f(x)}{h}$ is ^{≥ 0} ~~nonnegative~~ for negative h
~~nonpositive~~ ^{≤ 0} for positive h .
 Hence limit must be 0

Non Example if max occurs at boundary: $f(x) = x$ on $[0, 1]$

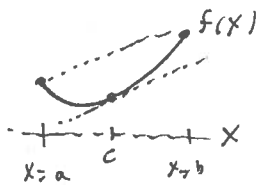


33) Mean value theorem:

Idea: For any secant line, there's a point with same slope



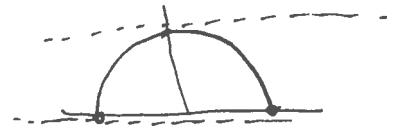
Precisely: If f continuous on $[a, b]$, differentiable on (a, b) , $\exists c$ st $f'(c) = \frac{f(b) - f(a)}{b - a}$



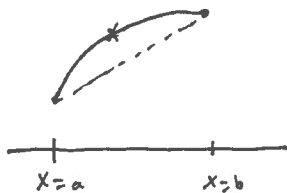
Non-example:

- not differentiable at one point.
no where has slope of secant line

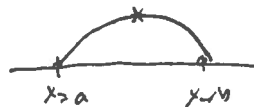
Note: f doesn't need to be differentiable on $[a, b]$,
MVT applies to $\sqrt{1-x^2}$ on $[-1, 1]$



Proof:



Subtract
away
secant
line



At max or min,
slope
must
be 0.

Application: Error bound in Taylor series

If f is smooth, $f(x) \approx f(0) + f'(0)x$

Let $e(x) = f(x) - f(0) - f'(0)x$. Claim: $|e(x)| \leq \max |f''| x^2$.

Proof:

$$\begin{aligned} e(x) &= e(x) - e(0) \\ &= e'(c_1) x && \text{by MVT} \\ &= [f'(c_1) - f'(0)] x \\ &= f''(c_2) c_1 x && \text{by MVT} \end{aligned}$$

$$\begin{aligned} |e(x)| &\leq \max |f''| |c_1| |x| \\ &\leq \max |f''| |x|^2 && \text{as } c_1 < x. \end{aligned}$$