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Analysis I
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Day 8 — Summary — Riemann Integration and Taylor Series

42. Darboux criterion: The function f is Riemann integrable on $[a, b]$ if and only if for all ε there is a partition P for which $U_a^b(f, P) - L_a^b(f, P) < \varepsilon$.

43. Continuous functions are Riemann integrable (on closed bounded domains).

44. The function f is Riemann integrable on $[a, b]$ with value s if and only if for all ε there is a δ such that $U_a^b(f, P) - s < \varepsilon$ and $s - L_a^b(f, P) < \varepsilon$ whenever $\|P_n\| < \delta$.

45. The Riemann integral has several inadequacies.

Exercises: Riemann integral DNE or ∞ or $-\infty$ or finite on $[0,1]$

a) $f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ \infty & \text{if } x = 1/2 \end{cases}$

b) $f(x) = \begin{cases} 1 & \text{if } x \leq 1/2 \\ \infty & \text{if } x > 1/2 \end{cases}$

c) $f(x) = \begin{cases} -\infty & \text{if } x \leq 1/3 \\ \infty & \text{if } x \geq 2/3 \\ 0 & \text{owise} \end{cases}$

42)

Proof

$\Rightarrow$: If f Riemann integrable $\exists$ P_1 st $U_a^b(f,P_1) - S < \varepsilon/2$

$\exists$ P_2 st $S - L_a^b(f,P_2) < \varepsilon/2$

Consider combinatn of P_1 & P_2. Call it P.

$U_a^b(f,P) - S < \varepsilon/2$ & $S - L_a^b(f,P) < \varepsilon/2$,

So $U_a^b(f,P) - L_a^b(f,P) < \varepsilon$

$\Leftarrow$: Suppose f not Riemann integrable.

~~$\sup_P L_a^b(f,P) < \inf_P U_a^b(f,P)$~~

$\inf_P U_a^b(f,P) - \sup_P L_a^b(f,P) \geqslant \varepsilon > 0$ for some ε.

So $\forall$ P_1, P_2 $U_a^b(f,P_1) - L_a^b(f,P_2) \geqslant \varepsilon$

Hence $\nexists$ P st $U_a^b(f,P) - L_a^b(f,P) < \varepsilon$ $\blacksquare$

43) Let $f \in C[a,b]$. f is Riemann integrable

Proof: By Darboux, suffices to show $\forall \varepsilon \, \exists P$ st $U(f,P) - L(f,P) < \varepsilon$

Fix ε.

As $f \in C[a,b]$, f is uniformly continuous.

Hence $\exists \, \delta$ st $|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{(b-a)}$

Consider a uniform partition of size $\frac{1}{n}$ where $\frac{1}{n} < \delta$.

On ~~each~~ i^{th} subinterval $M_i - m_i < \frac{\varepsilon}{(b-a)}$.

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} (M_i - m_i)\,\Delta x_i \leq \frac{\varepsilon}{b-a} \sum_{i=0}^{n-1} \Delta x_i = \frac{\varepsilon}{b-a}\, b-a = \varepsilon \quad \blacksquare$$