29 November 2014
Analysis I
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Coverpage to Pledged HW 13

Time limit: 3 hours. You may not use your books, your homeworks, your notes, or any electronics during the exam. Please write the start and finish times on your paper. Each subproblem is worth 10 points. To receive full credit, you must name all major theorems and state definitions used in your arguments. All counter examples must be accompanied by a proof. You may cite results from class and well-known theorems.

This homework is pledged. On the first page, please write your signature and the Rice University pledge: "On my honor, I have neither given nor received any unauthorized aid on this homework."

Due: Thursday, 4 December 2014 at the beginning of class.

[The exam is on the next page]

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1. (a) Is the countable intersection of compact sets compact? Prove it or find a counterexample.
   (b) Find a complete normed vector space V and a subset S such that S is closed and bounded but is not compact. Provide a direct proof that S is closed, bounded, and not compact.
2. Let $f : [a,b] \to \mathbb{R}$ be a real-valued function. Let $P = \{t_0 \leq t_1 \leq \ldots \leq t_n\}$ be a partition of $[a,b]$. Define the variation of f with respect to P to be $V_P(f) = \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|$. Define the total variation of f as $V(f) = \sup_P V_P(f)$.
   (a) Show that if f is non-decreasing and bounded, then $V(f)$ is finite.
   (b) Show that if $f \in C^1([a,b])$, then $V(f)$ is finite.
   (c) Is $V(f)$ a norm on $C^1([a,b])$? Prove your answer.
3. Let $f \in C^0([a,b])$.
   (a) Can f be approximated arbitrarily well (in a uniform sense) by a function in $C^2([a,b])$? Prove your answer.
   (b) Same, but for approximation in an ℓ_1 sense.
4. For any $\alpha = \{a_n\} \in \ell^1$, let $L(\alpha) = \sum_{n=1}^{\infty} a_n \cos nx$. Show that L is a continuous linear map of ℓ^1 (with the ℓ^1 norm) into $C^0(\mathbb{R})$ (with the sup norm).