25 August 2016
Analysis I
Paul E. Hand
hand@rice.edu

Week 2 — Summary — Differentiation, Mean Value Theorem, Taylor Series

23. The derivative of \( f \) at \( x \) is \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \), if this limit exists. A function is differentiable on a set if it is differentiable at every point in that set.

24. Product rule, quotient rule, chain rule.

25. Differentiability implies continuity.

26. *Let \( C^p([a,b]) \) be the set of functions defined on \([a,b]\) that are differentiable \( p \) times, and the \( p \)-th derivative is continuous. Let \( C^\infty \) be the set of functions that are in \( C^p \) for all \( p \).

27. At a local maximum (or minimum) of a differentiable function, the derivative is zero (provided that this max or min occurs in the interior of the function’s domain).

28. *Mean value theorem: If \( f \) is continuous on \([a,b]\) and is differentiable on \((a,b)\), then for some \( c \in (a,b) \),
   \[ f'(c) = \frac{f(b) - f(a)}{b-a}. \]

29. Big oh and Little oh notation:
   (a) \( f(x) = o(g(x)) \) as \( x \to x_0 \) means that \( f(x)/g(x) \to 0 \) as \( x \to x_0 \)
   (b) \( f(x) = O(g(x)) \) as \( x \to x_0 \) means that there exists \( C \) such that \( |f(x)| \leq Cg(x) \)

30. *A Taylor series is a local approximation of a function, and it is obtained by matching the value and a given number of derivatives of that function at a particular point.

31. *The \( n \)th order Taylor series of \( f(x) \) about \( x = a \) is given by
   \[ f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \]

32. The \( n \)th Taylor remainder term is
   \[ R_n(x) = f(x) - \left( f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right). \]

33. *The \( n \)th order Taylor series is accurate to the \( n + 1 \)st order in the neighborhood of the point of expansion. The constant factor of the error term is controlled by the maximum value of the \( n + 1 \)st derivative of the function.

   If \( f \in C^{n+1} \) in a neighborhood of \( a \), then \( R_n(x) = O(|x-a|^{n+1}) \) as \( x \to a \). More precisely,
   \[ R_n(x) \leq \max |f^{(n+1)}| \cdot \frac{|x-a|^{n+1}}{(n+1)!}. \]

   The max is taken over the neighborhood and the inequality holds for all points in the neighborhood.