Week 5 — Summary — Inner Products, Equivalent Norms, Complete Normed Vector Spaces

48. An inner product \( \langle \cdot, \cdot \rangle \) satisfies the following axioms for all \( u, v, w \in V \):
   
   (a) \( \langle v, w \rangle = \langle w, v \rangle \)
   
   (b) \( \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \)
   
   (c) If \( c \in \mathbb{R} \), \( \langle cv, w \rangle = c\langle v, w \rangle = \langle v, cw \rangle \)
   
   (d) \( \langle v, v \rangle \geq 0 \) \( \forall v \) and \( \langle v, v \rangle = 0 \Rightarrow v = 0 \).

49. Inner products induce a norm \( \| v \| = \sqrt{\langle v, v \rangle} \).

50. *Inner products satisfy the Cauchy-Schwarz inequality \( \langle v, w \rangle \leq \| v \| \| w \| \).

51. *Notes from Bill Symes on Dimension Theory (linear independence, basis, dimension). See website.

52. *Definition: Two norms \( \| \cdot \|_a \) and \( \| \cdot \|_b \) are equivalent on a vector space \( V \) if there exists \( c, C > 0 \) such that

\[
c \| x \|_b \leq \| x \|_a \leq C \| x \|_b \forall x \in V.
\]

53. *All norms on finite dimensional vectors spaces, e.g. \( \mathbb{R}^n \), are equivalent.

54. *In infinite dimensional vector spaces, some pairs of norms are not equivalent.

55. *Definition: A sequence \( x_n \) in a normed vector space is Cauchy if

\[
\forall \varepsilon \exists N \text{ such that } n, m \geq N \Rightarrow \| x_n - x_m \| < \varepsilon.
\]

56. *In a normed vector space, we say that \( x_n \) converges to \( x \) if \( \forall \varepsilon \exists N \) such that \( n \geq N \Rightarrow \| x_n - x \| < \varepsilon \). We write this as \( \lim_{n \to \infty} x_n = x \).

57. *Definition: A vector space is complete if any Cauchy sequence converges to an element in the set.

58. *Definition: A Banach space is a complete normed vector space.

59. *Definition: \( \mathbb{R}^n \) is a Banach space under the \( \ell_\infty \) norm. By equivalence of norms on finite dimensional spaces, it is a Banach space under any norm.