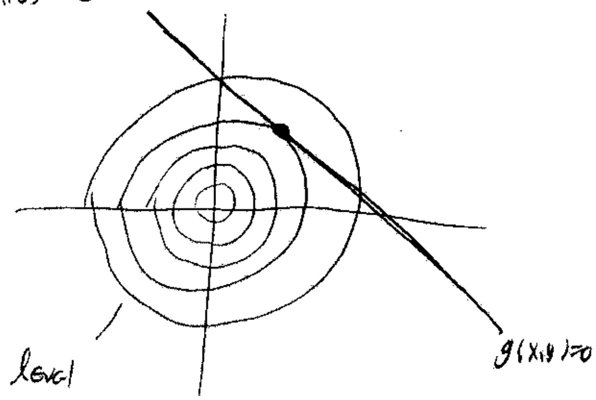
Justification of Lagrange Multipliers w/ equality constraint

max/min $f(x,y)$ st $g(x,y)=0$



$\mathcal{L} = f(x,y) + \lambda g(x,y)$

$\nabla \mathcal{L} = 0 \Rightarrow \nabla f(x,y) + \lambda \nabla g(x,y) = 0$

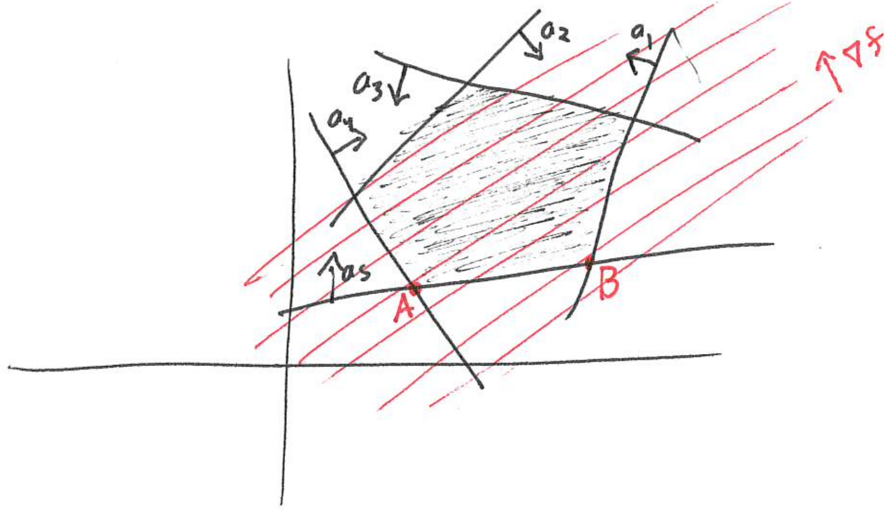
∇f is parallel to ∇g at constrained extremum

So level curves of f & g are tangent.

If level curves of f & g weren't tangent, you could move along constraint and increase OR decrease objective

Geometric Picture of Lagrange Multiplies for inequality constraints

$$\min_{x \in \mathbb{R}^n} f(x) \quad st \quad a_i \cdot x \geq b_i$$



$$\mathcal{L} = f(x) - \sum_i \lambda_i (a_i \cdot x - b_i)$$

$$\nabla f - \sum_i \lambda_i a_i = 0$$

$$\lambda_i \geq 0$$

$$\lambda_i (a_i \cdot x - b_i) = 0$$

Dual program of a convex program

Convex Program:

$$\min \; f(x) \quad \text{st} \quad \begin{array}{ll} g_i(x) \geq 0 & i=1\cdots m \\ h_i(x) = 0 & i=1\cdots p \end{array} \qquad \text{(primal)}$$

Lagrangian: $\mathcal{L}(x, \lambda, \nu) = f(x) - \lambda_i g_i(x) - \nu_i h_i(x)$

Lagrange dual function: $g(\lambda, \nu) = \inf_x \mathcal{L}(x, \lambda, \nu)$

Dual program: $$\max_{\lambda, \nu} \; g(\lambda, \nu) \quad \text{st} \quad \lambda_i \geq 0 \qquad \text{(dual)}$$

If there exist x^*, λ^*, ν^* st x^* feasible
λ^*, ν^* dual feasible
$f(x^*) = g(\lambda^*, \nu^*)$

then x^* is a minimizer of (primal).

(λ^*, ν^*) acts as a dual certificate. It certifies optimality of x^*.

KKT:

$$\nabla_x \mathcal{L}(x^*, \lambda^*, \nu^*) = 0 \quad \} \text{ optimality}$$

$$\cancel{\nabla_{\lambda,\nu} \mathcal{L}(x^*, \lambda^*, \nu^*) = 0} \quad g_i(x^*) \geq 0, \; h_i(x^*) = 0 \quad \} \text{ Feasibility}$$

$$\lambda_i^* \geq 0 \quad \} \text{ dual feasibility}$$

$$\lambda_i^* g_i(x) = 0 \quad \} \text{ complementary slackness}$$