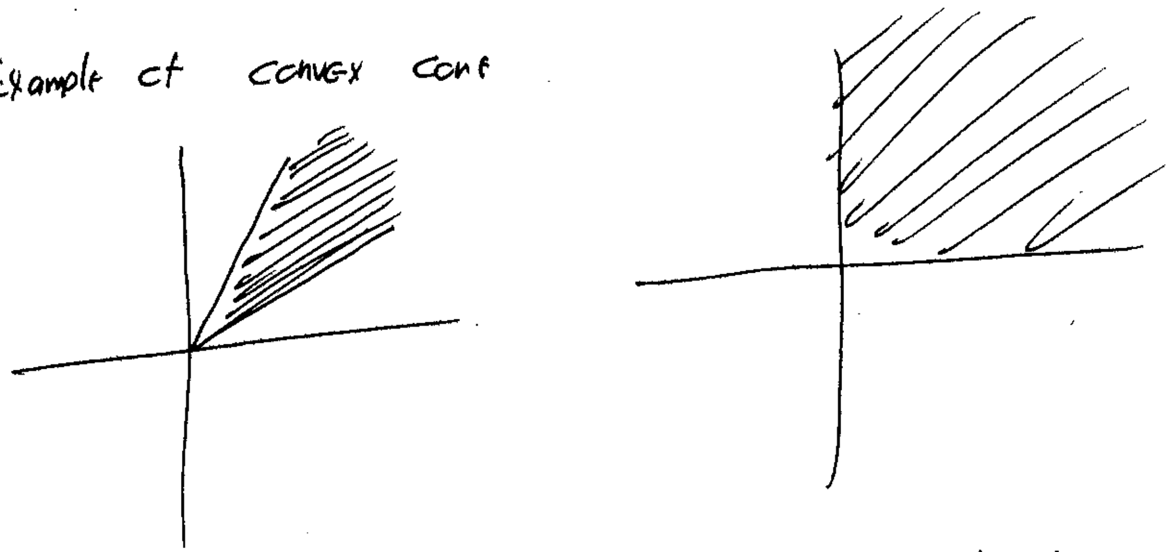
Cones

A set C is a cone if

$$\forall x \in C, \quad \theta x \in C \quad \forall \ \theta \in (0, \infty)$$

invariant to scale

Example of convex cone



A cone is proper if it is closed, convex, nonempty interior, and pointed ($x \in C$ & $-x \in C \Rightarrow x = 0$)

Examples:

positive orthant

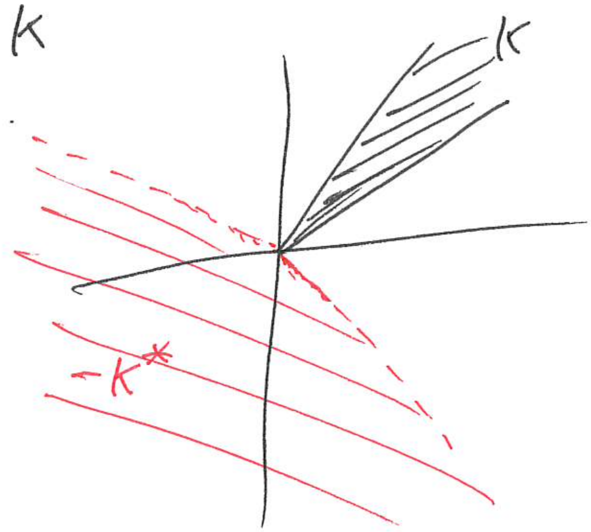
positive semidefinite matrices

Dual Cones

Let K be a cone:

$$K^* = \{ y \mid x^t y \geq 0 \;\; \forall x \in K \}$$

Geometrically, $y \in K^*$ iff y is normal to hyperplane supporting K



$K = \mathbb{R}^n_+$

$K = \{ \text{PSD } n \times n \text{ matrices} \}$

$\Big\}$ Self dual

Conic Inequalities

Given a proper cone, K,

write

$$X \preceq_K y \Rightarrow y - X \in K$$

Q is it true any matrix $\preceq_{PSD} cI$ for some c?

— — — — — $\succeq_{PSD} cI$ for some $c \neq 0$

Give two matrices A, B so that $A \not\succeq_{PSD} B$ and $B \not\succeq_{PSD} A$.

Convex Programs w/ generalized inequalities

$$\min \quad f_0(x) \quad st \quad \begin{array}{l} g_i(x) \succeq_{K_i} 0 \\ h_i(x) = 0 \end{array}$$

$$\mathcal{L} = f(x) - \sum_i \langle \lambda_i, g_i(x) \rangle - \sum_i v_i h_i(x)$$

$$g(\lambda, v) = \inf_x \mathcal{L}(x, \lambda, v)$$

Dual feasibility: $\lambda_i \succeq_{K_i^*} 0$

Dual program: $\max \; g(\lambda, v)$
$\quad st \;\; \lambda_i \succeq_{K_i^*} 0$

KKT:
$$g_i(x^*) \succeq_{K_i} 0$$
$$h_i(x^*) = 0$$
$$\lambda_i^* \succeq_{K_i} 0$$
$$\lambda_i^t g_i(x^*) = 0$$
$$\nabla_x \mathcal{L} = 0$$

Example of dual problem w/ PSD constraint

$$\min_{X \in S_{n\times n}} \langle I, X \rangle \quad \text{st} \quad \langle A_i, X \rangle = b_i. \quad X \succcurlyeq 0.$$

$$\mathcal{L} = \langle I, X \rangle - \sum_i \lambda_i (\langle A_i, X \rangle - b_i) - \langle Q, X \rangle$$

Dual feasibility $Q \succcurlyeq 0$

Comp slackness $\langle Q, X^* \rangle = 0$

KKT conditions at X^*

$$0 = I - \sum_i \lambda_i A_i - Q$$

$$X^* \succcurlyeq 0 \quad \langle A_i, X^* \rangle = b_i$$

$$Q \succcurlyeq 0$$

$$\langle Q, X^* \rangle = 0$$

Activity: If $Q \succcurlyeq 0$ and $Q_{11} = 0$, what can you say about Q?