Cones

A set \( C \) is a cone if
\[ \forall x \in C, \theta x \in C \quad \forall \theta \in (0, \infty) \]

invariant to scale

Example of convex cone

\[ \text{A cone is proper if it is closed, convex, nonempty interior,} \]
\[ \text{and pointed (} x \in C \land -x \in C \Rightarrow x = 0) \]

Examples:
- positive orthant
- positive semidefinite matrices
Dual Cones

Let $K$ be a cone:

$$K^* = \{ y \mid x^T y \geq 0 \implies y \in K \}$$

Geometrically, $y \in K^*$ iff $y$ is normal to hyperplane supporting $K$.

$$K = \mathbb{R}^n_+$$

$$K = \{ \text{PSD } n \times n \text{ matrices} \}$$

$\implies$ self-dual
Conic Inequalities

Given a proper cone, \( K \), write

\[ x \preceq_K y \implies y - x \in K \]

Is it true that any matrix \( \preceq_{\text{psd}} \) for some \( C^2 \)

\[ \preceq_{\text{psd}} c \cdot I \text{ for some } c > 0 \]

Give two matrices so that \( A \preceq_{\text{psd}} B \) and \( B \npreceq_{\text{psd}} A \).
Convex Programs w/ generalized inequalities

\[ \begin{align*}
\min \ f(x) \quad \text{st} \quad & \mathcal{G}_i(x) \preceq_{K_i} 0 \\
& h_i(x) = 0
\end{align*} \]

\[ L = f(x) - \sum_i \lambda_i \langle g_i(x), x \rangle - \sum_i \nu_i h_i(x) \]

\[ g(\lambda, \nu) = \inf_{x} L(x, \lambda, \nu) \]

Dual feasibility: \[ \lambda \preceq_{K_i} 0 \]

Dual program: \[ \max g(\lambda, \nu) \]
\[ \text{st} \quad \lambda \preceq_{K_i} 0 \]

KKT: \[ \begin{align*}
\mathcal{G}_i(x^*) & \preceq_{K_i} 0 \\
h_i(x^*) & = 0 \\
\lambda_i^* & \preceq_{K_i} 0 \\
\lambda_i^* \mathcal{G}_i(x^*) & = 0 \\
\nabla_x f & = 0
\end{align*} \]
Examples of dual problem w/ PSD constraint

\[ \min_{X} \langle I, X \rangle \text{ s.t. } \langle A_i, X \rangle = b_i, \]
\[ X \succeq 0. \]

\[ L = \langle I, X \rangle - \sum_i \lambda_i (\langle A_i, X \rangle - b_i) - \langle Q, X \rangle \]

Dual feasibility: \( Q \succeq 0 \)

Complementary slackness: \( \langle Q, X^* \rangle = 0 \)

KKT conditions at \( X^* \)

\[ 0 = I - \sum_i \lambda_i A_i - Q \]
\[ X^* \succeq 0 \quad \langle A_i, X^* \rangle = b_i \]
\[ Q \succeq 0 \]
\[ \langle Q, X^* \rangle = 0 \]

Activity: If \( Q \succeq 0 \) and \( Q_{11} = 0 \), what can you say about \( Q \)?