03/23/2017

**Theorem:** Let $A \in \mathbb{R}^{n\times n}$ and let $D_i$ be the closed disk centered at $a_{ii}$ with radius $r_i = \sum_{j\neq i} |a_{ij}|$:

$$D_i = \{z \mid |z - A_{ii}| \leq r_i\}$$

Then all eigenvalues of A lie in the union of the disks $D_i$, $i = 1, \dots, n$.

**Notation:** ~~Let~~ Let $A \in \mathbb{R}^{m\times n}$. Let $S \subseteq \{1, \dots, n\}$ with $|S| = p$. $A_S$ ~~are~~ is ~~$p\times p$~~ $m\times p$ matrix with the columns of A restricted to S.

~~Claim~~: Lemma: For any matrix A, $\operatorname{spark}(A) \geq 1 + \frac{1}{\mu(A)}$.

Claim: For any matrix A, if $(p-1)\mu(A) < 1$ then $\forall$ S with $|S| = p$, columns of $A_S$ are linearly independent.

**proof of claim:** Fix $S \subseteq \{1, \dots, n\}$ with $|S| = p$. Let $G = A_S^T A_S$. Note that

- $g_{ii} = 1 \quad \forall\ 1 \leq i \leq p$
- $|g_{ij}| \leq \mu(A) \quad \forall\ 1 \leq i,j \leq p,\ i \neq j$.

So, $\sum_{j\neq i} |g_{ij}| \leq (p-1)\mu(A) < 1 = g_{ii}, \quad \forall\ 1 \leq i \leq p$

$\Rightarrow$ G is a positive definite matrix by Greshgorin Circle theorem.

$\Rightarrow$ G columns of G are linearly independent.

Since $S$ is arbitrary, the result holds for all $S$ w/ $|S|=p$.

proof of lemma:

Fix $p = \text{spark}(A)$.

Assume $(p-1)\mu(A) < 1$

$\Rightarrow \forall S \subseteq \{1,\dots,m\}$ with $|S|=p$, columns of $A_S$ are linearly independant

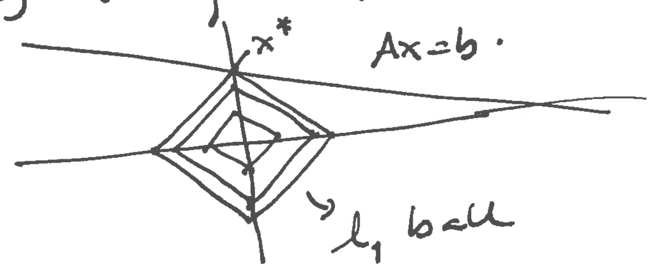
$\Rightarrow \text{spark}(A) > p$

contradiction

so, $(p-1)\mu(A) \geq 1 \Rightarrow p \geq 1 + 1/\mu(A)$

$\Rightarrow \text{spark}(A) \geq 1 + 1/\mu(A)$.

Why is $\min_x \|x\|_1$ s.t. $Ax=b$ likely to find sparse solution?

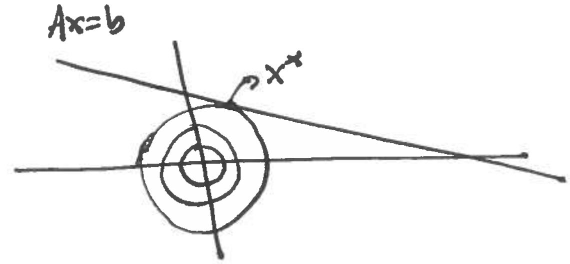



let $A = [1\ \ 4]$, $b = 4$

$x^* = \arg\min \|x\|_1$ s.t. $Ax=b$.

for $A \in \mathbb{R}^{1\times 2}$, with $A = [c, \pm c]$, $l_1$ minimization finds sparse solution.

Comparision to $l_2$ minimization.




Let $A = [1, 4]$, $b = 4$

$$x^* = \operatorname{argmin} \|x\|_2 \quad \text{s.t.} \quad Ax = b$$

for $A \in \mathbb{R}^{1\times 2}$ with $A = [0 \ \pm c]$ or $[\pm c \ \ 0]$, $l_2$ minimization finds sparse solutions.

<u>Null space property</u> (NSP)

<u>Definition</u>: A matrix $A$ satisfies null space property of order $K$ if there exist a constant $C > 0$ s.t.

$$\|h_S\|_2 \le C \frac{\|h_{S^c}\|_1}{\sqrt{K}}$$

~~$\forall$~~ holds $\forall\, h \in N(A)$ and for all ~~$A$~~ $S$ with ~~$\Lambda^c$~~ $|S| \le K$.

First consider a stronger definition of NSP, ~~(to distinguish~~

<u>Def$^n$</u> A matrix $A$ satisfies NSP of order $K$ if

$$\|h_S\|_1 < \|h_{\Lambda^c}\|_1$$

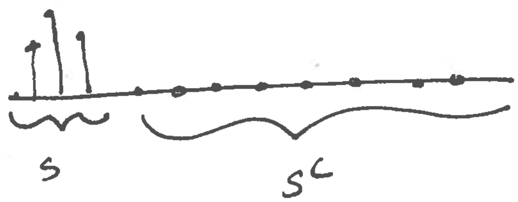
$\forall\ 0 \ne h \in N(A)$ and for all $S$ with $|S| \le K$.

**Thm:** $\forall\ x_0$ s.t. $\|x_0\|_0 \le K$, if $A$ satisfies NUP(K) then $x_0$ is the ! minimizer of

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = Ax_0. \qquad (*)$$

Note: If $x_0 + h$ is feasible then $Ah = 0 \Rightarrow h \in N(A)$.

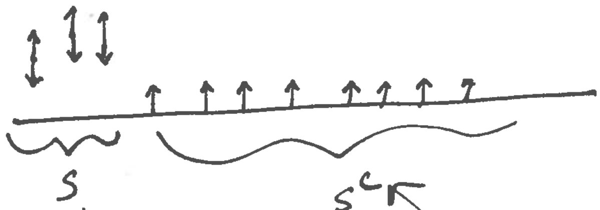
$x_0$ :




plot of absolute value.

Any nonzero h on $S^c$ increases $\ell_1$ norm.

$x_0 + h$ :




If $\|x_0 + h\| \le \|x_0\|_1$, these must go down by more than or equal to the amount these go up. ~~in absolute~~.

**proof:** Let $S = \text{supp}(x_0)$. Suppose $x_0 + h$ is a solution to $(*)$

with $\|x_0 + h\|_1 \le \|x_0\|_1$

$\Rightarrow$ ~~$\|x_0 + h\|_1$~~ $\|x_0 + h_S\|_1 + \|h_{S^c}\|_1 \le \|x_0\|_1$

$\Rightarrow$ $\|x_0\|_1 - \|h_S\|_1 + \|h_{S^c}\|_1 \le \|x_0\|_1$

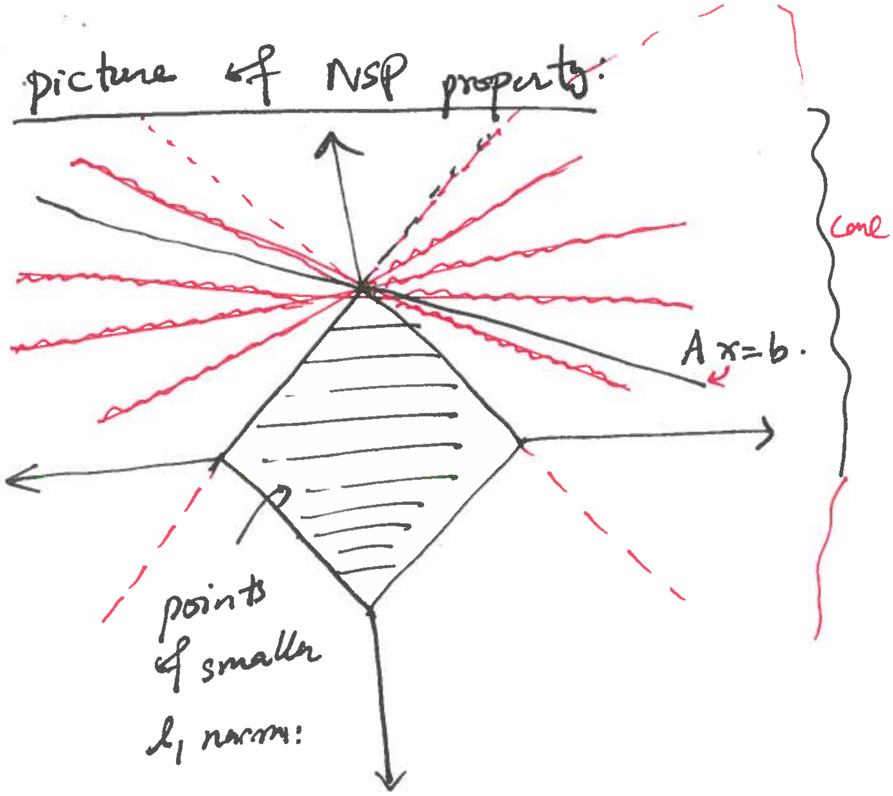
$\Rightarrow$ $\|h_{S^c}\|_1 \le \|h_S\|_1$

If NUP holds, $\|h_S\|_1 < \|h_{S^c}\|_1$ or $h = 0$

$\Rightarrow$ $h = 0$

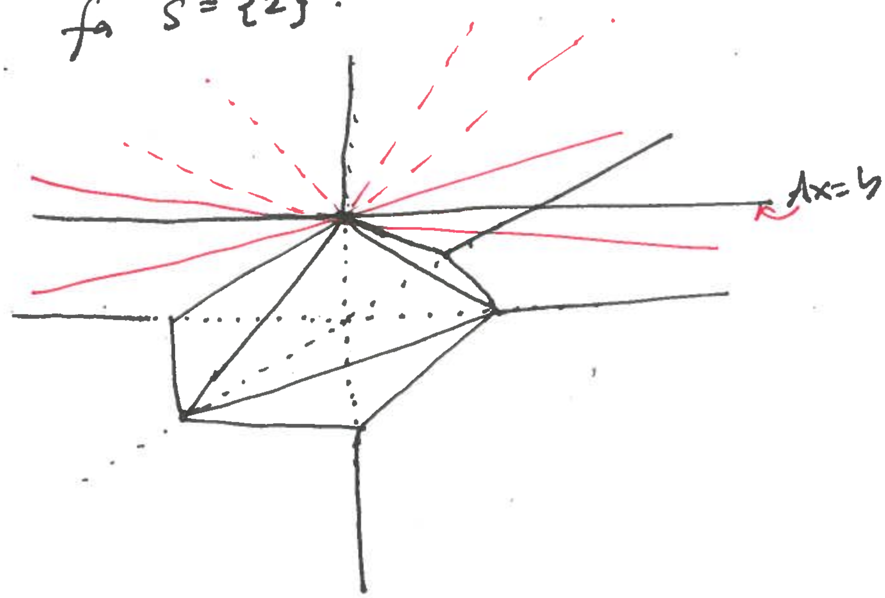
$\Rightarrow$ $x_0$ is ! minimizer of $(*)$.

picture of NSP property:



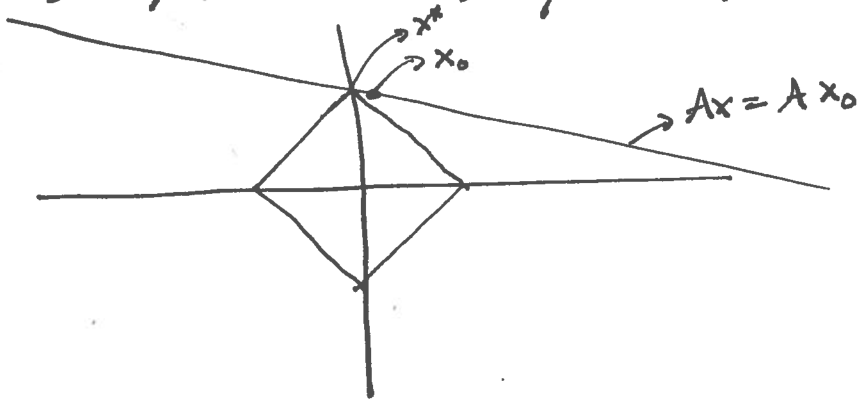

Null space condition then says affine solution sets live in this cone, which does not contain points of lower $\ell_1$ norm.

note that for $A = [1, 4]$, $b = 4$, $A$ satisfies $NUP(1)$ for $S = \{2\}$.




Null space condition says $Ax=b$ lives in the cone outside the octahedron.

Recovery of not exactly sparse signals.




__Claim:__ If $A$ has NSP$(K)$ with $C<1$, then ~~$x^* = \text{argmin}\|x\|$~~

$x^* = \text{argmin} \|x\|_1$ s.t. $Ax = Ax_0$ satisfies

$\|x^* - x_0\|_2 \leq C_1 \|x_0 - x_{0,K}\|_1$,

where $x_{0,K}$ is the top $K$ coefficient of $x_0$.

__proof:__ Let $x$ be a feasible point $\Rightarrow$ $\|x^*\|_1 \leq \|x\|_1$

Let $h = x^* - x_0$.

Let $S$ be the top $K$ coefficient of $h$.

Then ~~$\|h_{\Lambda^c}\|_1 \leq$~~

$\|h_{S^c}\|_1 \leq \|h_S\|_1 + 2\|x_0 - x_{0,K}\|_1$ (optimality)

Lemma 1.6 in Introduction to Compressed sensing by Davenport, Duarte, Eldar, Kutyniok.

By NSP:

$$\|h_S\|_2 \leq \frac{C\|h_{S^c}\|_1}{\sqrt{K}} \leq \frac{C}{\sqrt{K}}\|h_S\|_1 + \frac{2C}{\sqrt{K}}\|x_0 - x_{0,K}\|_1$$

$$\leq C\|h_S\|_2 + \frac{2C}{\sqrt{K}}\|x_0 - x_{0,K}\|_1$$

So, $$(1-C)\|h_S\|_2 \leq \frac{2C}{\sqrt{K}}\|x_0 - x_{0,K}\|_1$$

If $c < 1$,

$$\|h_S\|_2 \le \frac{2c}{(1-c)\sqrt{K}} \|x_0 - x_{0,K}\|_1 .$$

can improve this by $1/\sqrt{K}$ to get $\|h\|_2 \lesssim C' \frac{\|x_0 - x_{0,K}\|_1}{\sqrt{K}}$

So,
$$\begin{aligned}
\|x^* - x_0\|_2 = \|h\|_2 &\le \|h_S\|_2 + \|h_{S^c}\|_2 \\
&\le \|h_S\|_2 + \|h_{S^c}\|_1 \\
&\le \|h_S\|_2 + \|h_S\|_1 + 2\|x_0 - x_{0,K}\|_1 \\
&\le (1+\sqrt{K})\|h_S\|_2 + 2\|x_0 - x_{0,K}\|_1 \\
&\le \left[\frac{2c}{(1-c)\sqrt{K}}(1+\sqrt{K}) + 2\right] \|x_0 - x_{0,K}\|_1
\end{aligned}$$

$$\Rightarrow \|h\|_2 \lesssim C \|x_0 - x_{0,K}\|_1$$