Chi Squared Random Variables

A Chi-squared variable with $n$ d.o.f. is the sum of $n$ squared-Gaussians

If $Z \sim \chi^2_n$ then $Z = \sum_{i=1}^{n} X_i^2$ where $X_i \sim N(0,1)$

Activity: What is typical value or $\chi^2_n$ (for large $n$)?

What is typical variation?

If $X \sim \chi^2_1$, roughly $P(X > t)$ for large $t$?
Tail Bounds for Chi Squared Variables

\[ P( \chi_n^2 \leq n - 2\sqrt{n} t) \leq e^{-t^2} \]
\[ P( \chi_n^2 \geq n + 2\sqrt{n} t + 2t^2) \leq e^{-t^2} \]
\[ P( \chi_n^2 \geq n + \sqrt{n} t) \leq e^{-3t^2/16} \text{ for } 0 < t < \sqrt{n/2} \]
\[ P( \chi_n^2 > n(4t) ) \leq \frac{C}{\sqrt{n}} e^{-nt^2/2} \text{ for } 0 \leq t < 1 \]

Refs:
Laurent + Massart
Amini + Wainwright

Diagram:
- Gaussian Decay at \( \sqrt{n} \)
- Exponential decay at \( \frac{3n}{2} \)
- \( n \) axis
- \( \frac{3n}{2} \) axis
Length of random vectors

Let $X \in \mathbb{R}^n$ be st $X \sim N(0, I)$ (aka $X \sim N(0, I_{n \times n})$)

For some $c$,

$P\left( n(1-\varepsilon) \leq \|X\|_2^2 \leq n(1+\varepsilon) \right) \geq 1 - 2e^{-c\varepsilon^2 n}$ for $0 < \varepsilon < \frac{1}{2}$

Proof: Apply $P(\chi_n^2 \geq n + \sqrt{n} t) \leq e^{-\frac{2t^2}{n}}$ for $0 < \varepsilon \leq \frac{\sqrt{n}}{2}$

with $t = \sqrt{n} \varepsilon$.

Apply $P(\chi_n^2 \leq n - 2\sqrt{n} t) \leq e^{-t^2}$ with $t = \sqrt{n} \varepsilon \frac{\sqrt{n}}{2}$.

Meaning: In high dimensions, the length of a random vector is very highly certain.

Related: Most of the $n$-ball in high dims is located near the surface.
**Subexponential Random Variables**

A subexponential variable is one with a tail that is at most exponentially decaying.

A r.v. $X$ is subexponential if for some $K_1, K_2$

$$P(|X| > t) \leq e^{-t/K_1} \quad \forall t \geq 0$$

or

$$\left( \mathbb{E}[|X|^p] \right)^{1/p} \leq K_2 P \quad \forall P \geq 1$$

(These are equivalent)

The subexponential norm of $X$ is $\|X\|_{\psi_1} = \sup_{P \geq 1} P \left( \mathbb{E}[|X|^p] \right)^{1/p}$

**Lemma.** Let $X$ be subexponential, $\mathbb{E}[X] = 0$.

For $|t| \leq \sqrt{\|X\|_{\psi_1}}$, one has

$$|\mathbb{E} e^{tX}| \leq e^{C_1 t^2 \|X\|_{\psi_1}^2}$$

Here, $C_1$ are universal constants.
Proposition (5.16 in Vershynin) Bernstine Inequality

Let $X_1 \ldots X_N$ be independent, zero-mean, subexp. rvs and $K = \max_i \|X_i\|_{\psi_1}$. For every $t \geq 0$

$$P\left( \sum_{i=1}^{N} X_i \geq t \right) \leq 2 \exp\left[-c \min\left(\frac{t^2}{K^2 N}, \frac{t}{K}\right)\right]$$

Corollary

$$P\left( \left| \frac{1}{N} \sum_{i=1}^{N} X_i \right| \geq \varepsilon \right) \leq 2 \exp\left(-c \min\left(\frac{\varepsilon^2}{K^2 N}, \frac{\varepsilon}{K}\right)N\right)$$

Proof: WLOG, let $K = 1$.

Let $S = \sum_{i} X_i$. Let $\lambda > 0$

$$P\left(S \geq t\right) = P\left(e^{\lambda S} \geq e^{\lambda t}\right) \leq e^{-\lambda t} \mathbb{E}e^{\lambda S} = e^{-\lambda t} \prod_{i} \mathbb{E}[e^{\lambda X_i}]$$

$$\leq e^{-\lambda t} \prod_{i} \left(1 + \frac{\lambda^2}{4}\right) = e^{-\lambda t} \prod_{i} \left(1 + \lambda^2 \right)$$

Choose $\lambda = \min\left(\frac{t}{2N}, c\right)$ we have

$$P\left(S \geq t\right) \leq e^{-\exp\left[-\min\left(\frac{t^2}{4c N}, \frac{ct}{2}\right)\right]}$$

Prop. follows by $P(S \leq -t)$ by symmetry.

Corollary follows by taking $EN = t$. 