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CAAM 654
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Day 3 — Reading and Questions

Read: Dustin Mixon's blog post on 'A geometric intuition for the null space property' until it mentions RIP; Defn 1.2, Lemma 1.6, Thm 1.8 in Chapter 1 of Eldar and Kutyniok. This lemma and theorem involve the RIP, which we have not yet discussed. In problem 4 of this assignment, you will formulate at a variant of them involving the NSP.

1. Draw and explain a picture of why $\min \|x\|_1$ s.t. $Ax = b$ is likely to find something sparse where $\min \|x\|_2$ s.t. $Ax = b$ is likely to find something not sparse.

2. Notation: v_S is the restriction of the vector v to the coefficients in the set S. In Dustin Mixon's blog, an $m \times n$ matrix A is said to have the null space property of order s, NUP(s), if $\forall v \in \mathcal{N}(A) \setminus \{0\}$, $\|v_S\|_1 < \|v_{S^c}\|_1$ for all subsets $S \subset \{1, \dots, n\}$ with cardinality at most s. Write out the gist of the reasoning why A satisfies NUP(s) if and only if $\forall \|x_0\| < S$, x_0 is the unique minimizer of $\min \|x\|_1$ s.t. $Ax = Ax_0$.

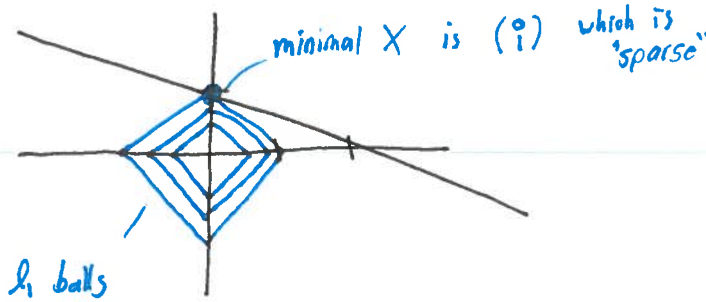
3. The book defines the null space property of order k, NSP, differently than above. Is the books version stronger or weaker? Prove it. In what senses are each definition better than the other?

4. Write out a specialization of Lemma 1.6 and Theorem 1.8 for the case when we only know a signal has NSP of order $2k$ with constant C. Do this in the case where there is no signal noise and the signal is not necessarily exactly sparse.

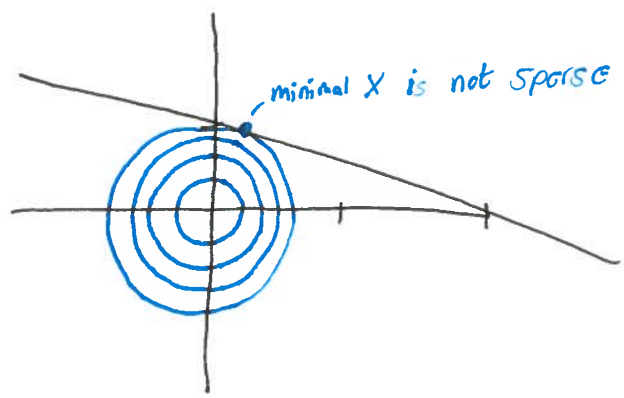
1) Why is $\min_x \|X\|_1$ st $Ax=b$ likely to find something sparse, while $\min_x \|X\|_2$ st $Ax=b$ is likely to find something dense?

Consider an example: $A=(1\ \ 2)$ $b=2$

$\min \|X\|_1$ st $Ax=b$



$\min \|X\|_2$ st $Ax=b$



min l_1 works at finding sparse soln when $A \in \mathbb{R}^{1\times 2}$ when $A \neq (c, \pm c)$

min l_2 — — — — — — — wln $A = (c, 0)$ or $(0, c)$.

2) What is connection of NUP. to signal recovery,

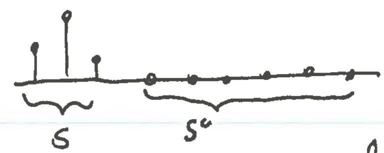
where NUP(S) means $\|V_S\|_1 < \|V_{S^c}\|_1$ of $\forall V \in N(A) \Rightarrow$ $\forall\ |S| < s$.

Signal recovery: $\forall x_0$, x_0 is ! minimizer of $\min \|x\|_1$ st $Ax = Ax_0$ (*)

Connection: Suppose $x_0 + h$ is feasible and $\|x_0+h\|_1 \leqslant \|x_0\|_1$.

$h \in N(A)$

x_0:



Any nonzero h on S^c increases l_1 norm.

$x_0 + h$:

these must go down by more than those go up in order to have smaller l_1 norm than x_0

Formally: Use triangle inequality and separability of l_1 norm

$\|x\|_1 = \|x_S\|_1 + \|x_{S^c}\|_1$

x restricted to indices S

Let $S = \text{support}(x_0)$. Suppose x_0+h is a soln to (*)

$\|x_0 + h\|_1 \leqslant \|x_0\|_1$

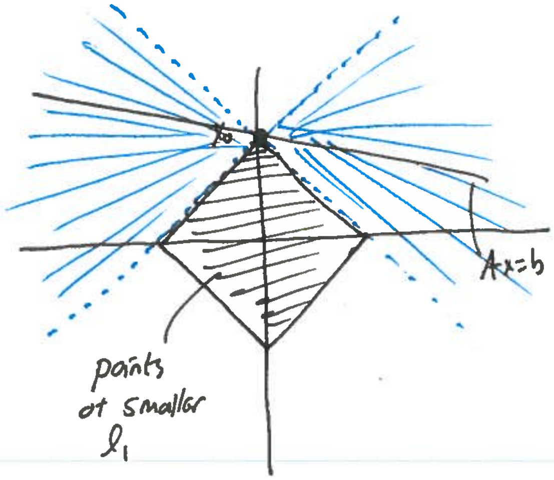
$\Rightarrow \|x_0 + h_S\|_1 + \|h_{S^c}\|_1 \leq \|x_0\|_1$

$\Rightarrow \|x_0\| - \|h_S\|_1 + \|h_{S^c}\|_1 \leq \|x_0\|_1$

$\Rightarrow \|h_{S^c}\|_1 \leqslant \|h_S\|_1$

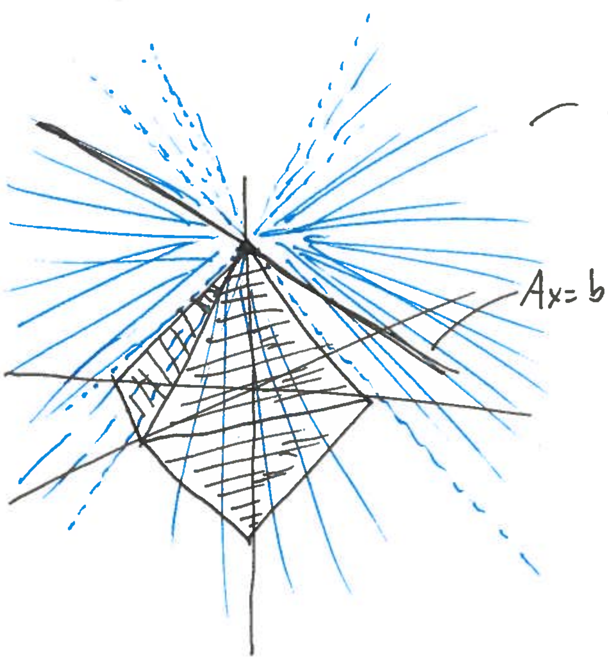
If NUP holds, $\|h_S\|_1 < \|h_{S^c}\|_1$ or $h = 0$. So $h = 0$.

Picture of Null space property



Null space condition says affine soln set lives in this cone, which does not contain points of lower ℓ_1 norm

In 2d.



null space condition says $Ax=b$ lives in this cone outside the octohedron