Optimization of Naghdi Shells

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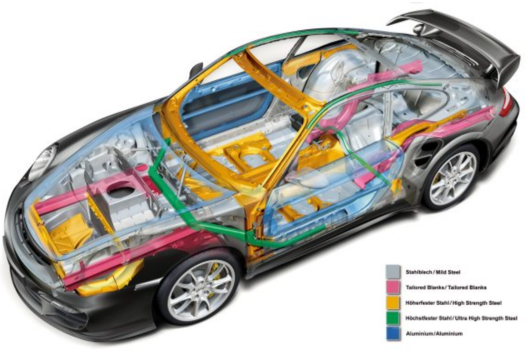
Outline

Shells in Nature

- Shell structures are ubiquitous in nature because they are light but strong.
- Geometry allows forces to be balanced by tensile strains rather than bending.

Manmade Shells



- For the 2008 Porsche 911 GT2, mechanical characteristics are of primary importance; optimization is highly desirable in engineering practice.
- For this 1707 Stradivarius, it is the acoustic output that matters most; bending is desirable.

Outline

Introduction
    Motivation

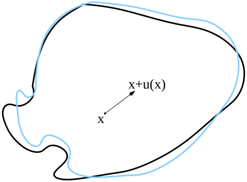
The Naghdi Shell Model
    Preliminaries
    Derivation
    Finite Element Discretization

Shape Optimization
    Model Optimization Problems
    Optimization Approach
    Numerical Results and Interpretation

Conclusion

3d Elasticity

In 3d elasticity, the undeformed body occupies the region Ω. Under deformation, the point x moves to $x + u(x)$.



Let $\Gamma_0 \cup \Gamma_1$ be a partition of $\partial\Omega$ with $\text{meas}(\Gamma_0) > 0$. The equilibrium equations of 3d elasticity are:

$$\begin{aligned} -\operatorname{div}\sigma &= f && \text{in } \Omega \text{ (force balance)} \\ \sigma &= H : e(u) && \text{in } \Omega \text{ (constitutive)} \\ u &= 0 && \text{on } \Gamma_0 \text{ (clamping)} \\ \sigma \cdot n &= h && \text{on } \Gamma_1 \text{ (boundary traction)} \end{aligned}$$

3d Elasticity

The corresponding weak form is to find

$$u \in \mathcal{U}_0 \equiv \left\{ u \in H^1(\Omega)^3 \ : \ u = 0 \text{ on } \Gamma_0 \right\}$$

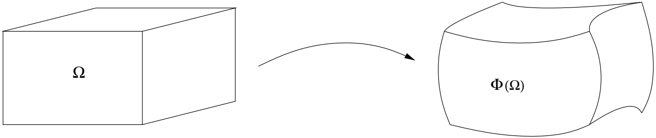
such that

$$\int_\Omega (H(x) : e(u)(x)) : e(v)(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx + \int_{\Gamma_1} h(x) \cdot v(x) \, d\Gamma$$

for all $v \in \mathcal{U}_0$.

- The strain tensor is $e(u) \equiv (\nabla u(x) + \nabla u(x)^T)/2$.
- The : operator represents tensor contraction.

The Chart Function

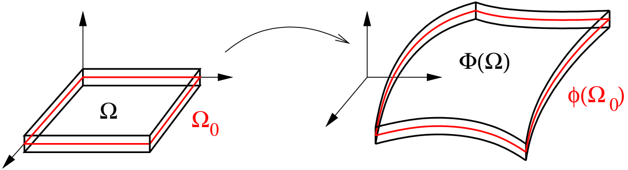


- Convenient to represent physical region (at right) with "reference coordinates".
- Mapping $\Phi : \Omega \to \mathbb{R}^3$ is injective, and $(\nabla\Phi)^{-1}$ exists $\forall x \in \Omega$, i.e., the derivatives of Φ form a basis.
- Dual bases are defined via

$$\begin{aligned} g_i &= \partial_i \Phi, \\ g^i \cdot g_j &= \delta^i_j. \end{aligned}$$

- These bases can be used to represent vectors and tensors.

The Naghdi Shell Model (Geometry)



Geometry determined by

- Middle surface chart $\phi : \Omega_0 \subset \mathbb{R}^2 \to \mathbb{R}^3$. The normal vector is $a_3 = (a_1 \times a_2)/|a_1 \times a_2|$, with $a_\alpha = \partial_\alpha \phi$.
- Thickness function $t : \Omega_0 \to \mathbb{R}^+$,

through the mapping

$$\Phi(x_1, x_2, x_3) \equiv \phi(x_1, x_2) + x_3 a_3(x_1, x_2)$$

of the reference coordinates

$$\Omega \equiv \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega_0 \text{ and } |x_3| < t(x_1, x_2)/2\}.$$

The Naghdi Shell Model

In curvilinear coordinates, the weak form for 3d elasticity (without boundary traction) appears

$$\int_\Omega H^{ijkl} e_{ij}(u) e_{kl}(v) \sqrt{g} dx = \int_\Omega f(x) \cdot v(x) \sqrt{g} dx$$

- Under the Naghdi model, u is determined by a middle surface displacement $z : \Omega_0 \to \mathbb{R}^3$ and a rotation vector $\theta : \Omega_0 \to \mathbb{R}^3$ with $\theta \cdot a_3 = 0$:

$$u(x_1, x_2, x_3) = \underbrace{z(x_1, x_2)}_{\text{displacement}} + x_3 \underbrace{\theta(x_1, x_2)}_{\text{rotation}}.$$

- Integrate over x_3: assumed **kinematics** → **2d problem**.
- Additionally, assumes that $\sigma_{33} = 0$.

The Naghdi Shell Model

The Naghdi problem is to find $(\theta, z) \in \mathcal{R} \times \mathcal{V}$ such that for all $(\eta, y) \in \mathcal{R} \times \mathcal{V}$,

$$A(\theta, z; \eta, y) = \int_{\Omega_0} p_3 y_3 \sqrt{a}\, dx.$$

- The Naghdi bilinear form A is

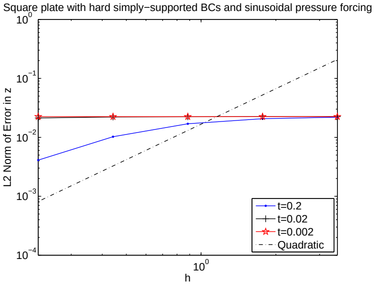
$$A(\theta, z; \eta, y) = \int_{\Omega_0} \Bigg(\tilde{C}^{\alpha\beta\lambda\mu} \left[t\gamma_{\alpha\beta}(z)\gamma_{\lambda\mu}(y) + \frac{t^3}{12}\chi_{\alpha\beta}(\theta, z)\chi_{\lambda\mu}(\eta, y) \right] + t\tilde{D}^{\lambda\mu}\zeta_\lambda(\theta, z)\zeta_\mu(\eta, y) \Bigg) \sqrt{a}\, dx.$$

  The tensors γ, χ, ζ describe respectively membrane stretching, bending, and transverse shear.

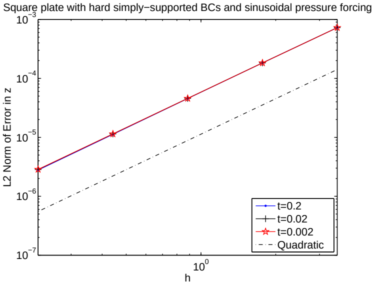
- Assuming that $\mathcal{R} \subset \left\{ \theta \in H^1(\Omega_0)^3 \ : \ \theta \cdot a_3 = 0 \text{ a.e.} \right\}$, and $\mathcal{V} \subset H^1(\Omega_0)^3$ prohibit rigid-body motions, this problem has a unique solution.

Locking in Shells

Naïve FEM



MITC3 Duran-Liberman



- These show L^2 error for the same problem with uniform mesh refinement for two FEM methods with the same number of degrees of freedom, but slightly different discretizations.
- Convergence non-uniform in t.
- Mesh refinement is not a practical strategy.

Asymptotics and Locking

To understand locking, must consider asymptotic behavior for thin shells.

- For $t(x) \equiv \mathrm{const.}$, and $U = (\theta, z) \in \mathcal{U} \equiv \mathcal{R} \times \mathcal{V}$, $V = (\eta, y) \in \mathcal{U}$, the Naghdi bilinear form can be written

$$A(U;V) = t^3 \underbrace{A_b(U;V)}_{\text{bending}} + t \underbrace{A_m(U;V)}_{\text{membrane}}.$$

- Key are the *pure-bending subspace*

$$\mathcal{U}_0 = \{U \in \mathcal{U} \ : \ A_m(U,U) = 0\},$$

  and the energy minimization problem

$$\min_{U \in \mathcal{U}} \frac{t^3}{2} A_b(U,U) + \frac{t}{2} A_m(U,U) - F(U).$$

- If pure bending is possible ($\mathcal{U}_0 \neq \emptyset$), as $t \to 0$, solution found by minimizing over $\mathcal{U}_0$.
- For a naïve discretization with piecewise polynomial functions, it can be shown that

$$\mathcal{U}_0 \cap \mathcal{U}_h = \{0\},$$

  and it is said that the finite elements "lock."

Locking Summary

Naïve FEM

$$\begin{tikzcd}
U_h^t \arrow[r, "h \to 0"] \arrow[d, "t \to 0"'] & U^t \arrow[d, "t \to 0"] \\
0 & U^0
\end{tikzcd}$$

Desired FEM

$$\begin{tikzcd}
U_h^t \arrow[r, "h \to 0"] \arrow[d, "t \to 0"'] & U^t \arrow[d, "t \to 0"] \\
U_h^0 \arrow[r, "h \to 0"] & U^0
\end{tikzcd}$$

- U_t^h is the solution at mesh size h to the problem with thickness t.
- Although even a locking formulation converges to U^t for fixed h, the diagram does not commute.

Outline

General Problem Form

- Optimization variables are the chart function ϕ and thickness t:

$$(\phi, t) \equiv g \in \mathcal{G}.$$

- Objective function of the form $j(g, U)$, which is some measure of deflection under load.
- Unique solution $U[g]$ given by solution to $c(g, U) = 0 \in \mathcal{U}'$, with

$$\langle c(g, U), V \rangle_{\mathcal{U}' \times \mathcal{U}} = A(U; V; g) - F(V; g).$$

The optimization problem is

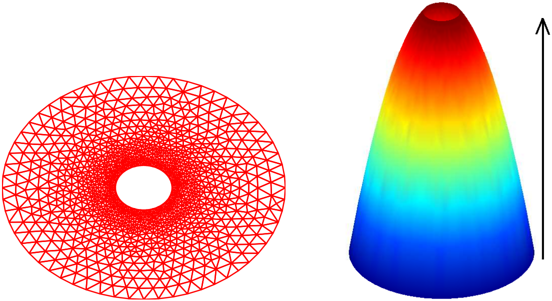
$$\min_{g \in \mathcal{G}} J(g) \equiv j(g, U[g]).$$

Shell Optimization in Practice

- For plates, there are FEM pairs that satisfy the discrete $\inf$-$\sup$ condition needed for convergence; for shells, none are known.
- Shell FEM based on plate FEM; $\inf$-$\sup$ condition tested by solving eigenvalue problems.
- Much of the optimization done in engineering literature; usually just pick FEM, mesh, “run it.”

Now, we show two simple model problems where the objective functions can easily be visualized.

Parabolic Shell of Revolution



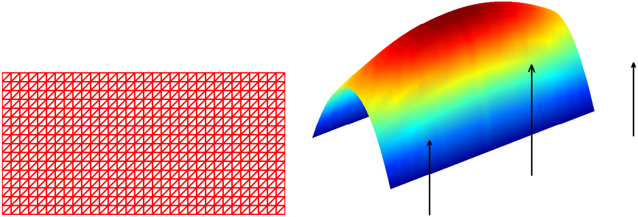
- Inner and outer radii r=10cm, R=50cm, t=0.1cm; subjected to crown load $q = 1.6 \cdot 10^{-3}\mathrm{kN}/\mathrm{cm}^2$ along inner edge.
- Design variable is the structure height h:

$$\phi(x_1, x_2) = (x_1, x_2, h(25/24 - (x_1^2 + x_2^2)/2400)).$$

- Objective function is crown deflection

$$j(g, U) = \frac{1}{20\pi} \int -e_3 \cdot z \, dx$$

Roof Shell with two Parabolic Generators



- Defined on $[-6\mathrm{m}, 6\mathrm{m}] \times [-3\mathrm{m}, 3\mathrm{m}]$, t=0.1m. Gravity load (design-dependent).
- Design variables are the parabola heights at the edges (s_1) and the center (s_2).

$$\phi(x_1, x_2) = \left(x_1, x_2, \left(\frac{s_1 - s_2}{36} x_1^2 + s_2 \right) \left(-\frac{x_2^2}{9} + 1 \right) \right).$$

- Objective function is the strain energy

$$j(g, U) = \frac{1}{2} A_g(U, U).$$

From Camprubí, Bischoff and Bletzinger (2004).

Derivative Computation

Need derivatives of the objective function,

$$DJ(g) = D_g j(g, U[g]) + D_U j(g, U[g]) DU[g].$$

FEM approximation of $DJ(g)$ requires two linear system solves:

1. Solve the state equation $c(g, U) = 0$ for $U[g]$.
2. Solve the adjoint equation

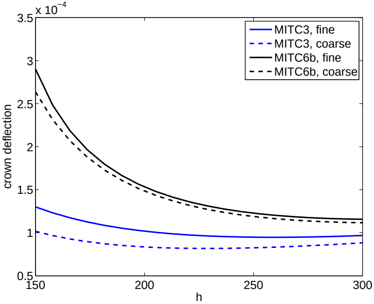
   $$\langle D_u c^{\times}(g, U[g]) P, V \rangle_{\mathcal{U}' \times \mathcal{U}} = \langle -D_U j(g, U[g]), V \rangle_{\mathcal{U}' \times \mathcal{U}}$$

   for the adjoint state P. Easy: this is just another shell equation with loading determined by the objective function.
3. Compute the gradient via

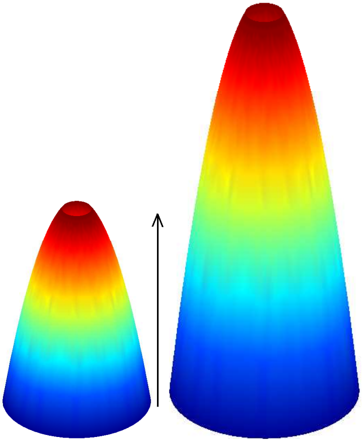
   $$\langle DJ(g), \delta g \rangle_{\mathcal{G}' \times \mathcal{G}} = \langle D_g j(g, U[g]), \delta g \rangle_{\mathcal{G}' \times \mathcal{G}} + \langle D_g c(g, U[g]) \delta g, P \rangle_{\mathcal{U}' \times \mathcal{U}}.$$

   Harder: requires shape derivatives of shell operators.

Parabolic Shell of Revolution

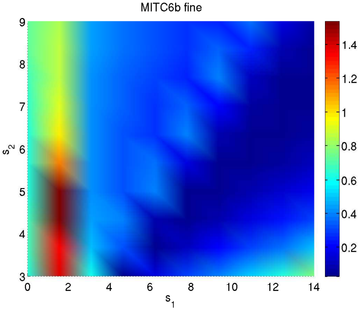


- Objective evaluated with MITC3 Duran-Liberman shell elements, checked against MITC6b shell elements.
- Important to resolve boundary layer around crown.
- Locking shifts optimal solution to the left.

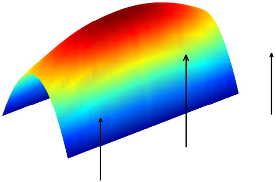


Inital shape $h = 150$, and optimal shape (MITC3 coarse), $h \approx 230$.

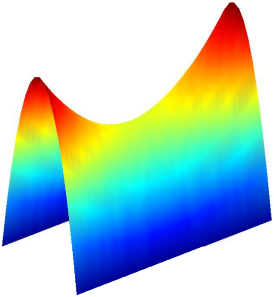
Roof Shell with two Parabolic Generators



- Objective evaluated with MITC6b shell elements.
- Taller makes it heavier.
- $s_1 > s_2$ means less bending.



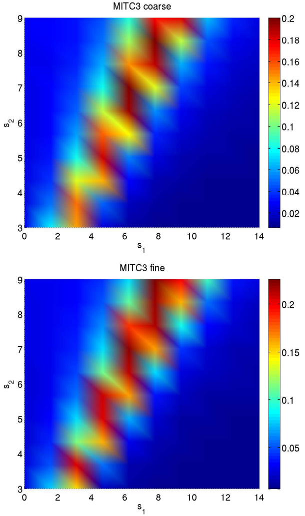
Inital shape $(s = [4, 6])$.



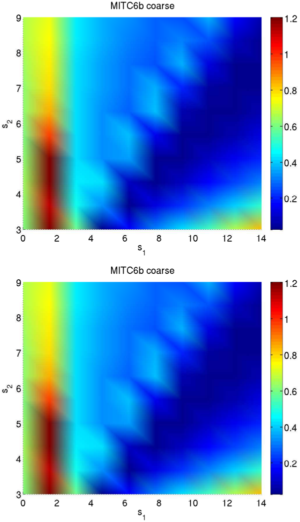
Optimal shape, $(s \approx [10.9, 6.3])$.

MITC3 vs. MITC6b for Roof Shell

MITC3 Duran-Liberman



MITC6b



- MITC6b results appear to have converged.
- MITC3 gives substantially different results.

Conclusions

- Locking is a central issue in the optimization of shell structures: optimization incorrectly favors bending.
- Have implemented MITC3 shell elements with Duran-Liberman modification, for shell equations and calculation of shape derivatives.
- Difficult to account for differences between different formulations.
- Issues of chart function smoothness, representation of the geometry need more work.
- Examples demonstrate utility of simple problems, importance of resolving boundary layers.