Preconditioners for Karush–Kuhn–Tucker Matrices Arising in the Optimal Control of Distributed Systems

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Abstract

In this paper preconditioners for linear systems arising in interior–point methods for the solution of distributed control problems are derived and analyzed. The matrices K in these systems have a block structure with blocks obtained from the discretization of the objective function and the governing differential equation. The preconditioners have a block structure with blocks being composed of preconditioners for the subblocks of the system matrix K. The effectiveness of the preconditioners is analyzed and numerical examples for an elliptic model problem are shown.

Key words and phrases. Preconditioners, iterative methods, interior point methods, linear quadratic optimal control problems.

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1 Introduction

The discretization of distributed linear quadratic optimal control problems with bound constraints on the controls and on the states leads to large scale quadratic programming problems. Because of their complexity and convergence properties, interior point methods are attractive solvers for such problems. They are iterative methods which in each iteration generate approximations to solutions that are strictly feasible with respect to the bound constraints. Within each iteration, large indefinite linear systems have to be solved. If interior point methods are applied to linear quadratic control problems governed by partial differential equations, then iterative techniques usually have to be applied to solve these linear systems. To make interior point methods efficient, it is important to solve these linear systems efficiently. Krylov subspace methods are iterative linear system solvers, which are very suitable in this context. They do not require the system matrix in explicit form, but only require matrix vector multiplications. This is very useful since for the problems under investigation the system matrices have a block structure in which blocks are related to discretized differential equations. The convergence of Krylov subspace methods depends on the distribution of the eigenvalues of the system matrix. Roughly speaking, their convergence is the better the more the eigenvalues of the system matrix are clustered and the smaller the clusters are. Ill–conditioning of the matrix, i.e. a large quotient of largest absolute eigenvalue divided by smallest absolute eigenvalue, typically

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corresponds to a poor convergence of Krylov subspace methods. To improve the convergence of these methods nonsingular matrices are constructed so that the similarity transformation with these matrices leads to a system with better clustered eigenvalues. These matrices are called preconditioners. The purpose of this paper is the construction of such preconditioners for systems arising in interior–point methods for certain distributed control problems.

To illustrate the issues, we consider the following elliptic model problem.

$$\min \frac{1}{2}\int_{\Omega}(y(x)-y_d(x))^2 + \frac{\gamma}{2}\int_{\partial\Omega} u^2(x)ds \tag{1.1}$$

over all (y, u) satisfying the state equation

$$\begin{aligned} -\Delta y(x) + y(x) &= f(x) \qquad x \in \Omega, \\ \tfrac{\partial}{\partial n} y(x) &= u(x) \qquad x \in \partial\Omega, \end{aligned} \tag{1.2}$$

and the bound constraints

$$\begin{aligned} y_{\text{low}} \le y(x) \le y_{\text{upp}} \quad \text{a.e.}, \\ u_{\text{low}} \le u(x) \le u_{\text{upp}} \quad \text{a.e.} \end{aligned} \tag{1.3}$$

A discretization of the problem with, say, finite elements, leads to a quadratic programming problem of the form

$$\min \frac{1}{2} y_h^T M_y y_h + \frac{\gamma}{2} u_h^T M_u u_h + c^T y_h + d^T u_h \tag{1.4}$$

subject to

$$A y_h + B u_h = b, \tag{1.5}$$

$$\begin{aligned} y_{\text{h,low}} \le y_h \le y_{\text{h,upp}}, \\ u_{\text{h,low}} \le u_h \le u_{\text{h,upp}}. \end{aligned} \tag{1.6}$$

Here h indicates the mesh size of the discretization and $u_h \in I\!R^{n_u}$, $y_h \in I\!R^{n_y}$ represent the discretized controls and states, respectively. The matrices $M_u \in I\!R^{n_u \times n_u}$ and $M_y \in I\!R^{n_y \times n_y}$ are positive definite. The vectors $y_{\text{h,low}}, \ldots, u_{\text{h,upp}}$ are obtained from the bound constraints (1.3) in a straightforward way.

There are various classes of interior point methods. They all (after possible transformations) require the solution of linear systems with system matrices

$$K = \begin{pmatrix} H_y & 0 & A^T \\ 0 & H_u & B^T \\ A & B & 0 \end{pmatrix}, \tag{1.7}$$

where

$$H_y = M_y + D_y, \quad H_u = \gamma M_u + D_u, \tag{1.8}$$

with some positive semidefinite diagonal matrices D_y and D_u. Since the matrix K is related to matrices arising in the Karush–Kuhn–Tucker optimality conditions, we call K a Karush–Kuhn–Tucker (KKT) matrix.

Even though the exact form of the diagonal matrices D_y and D_u differs from interior point method to interior point method, they all have in common that diagonals of D_y and D_u grow unbounded if the corresponding components of y_h or u_h converge towards a bound.

The matrices K arising in interior–point methods for the solution of problems like (1.1)–(1.3) are usually ill–conditioned. There are at least two sources for the ill-conditioning. One source is the discretization of the infinite dimensional problems. Typically, the eigenvalues of K spread out towards zero if the discretization is refined. The second source are the large diagonals in D_y and D_u that arise if variables approach the bound. This source is due to the interior–point method. Ill–conditioning also arises if the original infinite dimensional problem is ill–posed. The preconditioners derived in this paper are designed to remedy the ill–conditioning arising from the first two sources. They use the block structure of K and are composed of preconditioners for the blocks M_y, M_u, and A of K. This allows the use of known preconditioners for the governing differential equations. Moreover, computationally expensive parts of the preconditioner have to be computed only once during the interior–point method, since only the diagonal contributions D_y and D_u change from one interior–point iteration to another.

Preconditioners for problems related to this one are investigated in other papers. There are several papers, e.g. [5], [15], [17], investigating preconditioners for systems arising in the numerical solution of partial differential equations such as the Stokes equations, or the biharmonic equation. These systems can also be viewed as KKT systems. However, the blocks in those matrices are different and, therefore, the preconditioners for those problems are different than the ones introduced here. In fact, if the governing equations would be the Stokes equations, or the biharmonic equation, then the preconditioners in the papers cited above could be used as blocks in the preconditioners introduced here. Some of the tools provided in those papers, in particular a result from [15], cf. Lemma 5.1, are heavily used in our analysis. Preconditioners for interior–point methods for linear programs (LP) are investigated in [9], [10]. Those preconditioners are for general LPs and are based on sparse matrix factorizations or on the SOR method. Since no particular structure is assumed, those papers do not contain any theoretical result on the quality of the preconditioner.

This paper is organized as follows. In the first part we study the QP problem. Section 2 investigates the problem (1.1)–(1.3) and its discretization. The Sections 3 and 4 discuss the optimality conditions for the QP (1.4)–(1.6) and some aspects of interior–point methods relevant for the construction of preconditioners. Section 5 contains some essential results about the Krylov subspace methods MINRES and SYMMLQ. The preconditioners are introduced and analyzed in Section 6. This section also contains some numerical tests demonstrating the quality of the preconditioners.

2 The Control Problem

As noted in the introduction, one source of ill–conditioning in the KKT matrix is the discretization of the infinite dimensional problem. This section provides some results needed to address this aspect of the problem. These results can be proven for a general class of problems, which include the model problem (1.1)–(1.3) as a special case. In this section we do not consider the control or the state constraints.

2.1 The Abstract Problem

Let $\mathcal{Y}$ and $\mathcal{U}$ be Hilbert spaces. These spaces play the role of the state and the control space, respectively. Moreover, let a, b be continuous bilinear forms on $\mathcal{Y} \times \mathcal{Y}$ and $\mathcal{U} \times \mathcal{Y}$, respectively. In addition, we assume that a is $\mathcal{Y}$–elliptic. In particular, there exist constants $\alpha > 0$ and $\beta > 0$ with

$$\alpha \|y\|_{\mathcal{Y}}^2 \le a(y,y), \quad b(u,y) \le \beta \|u\|_{\mathcal{U}} \|y\|_{\mathcal{Y}}, \quad \forall y \in \mathcal{Y},\ u \in \mathcal{U}.$$

Furthermore, let $\mathcal{Z}$ be a Hilbert space and $\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$. In particular there exists $\zeta > 0$ such that

$$\|\mathcal{C}y\|_{\mathcal{Z}} \le \zeta \|y\|_{\mathcal{Y}} \quad \forall y \in \mathcal{Y}.$$

With some linear functional l on $\mathcal{Y}$ we consider the problem

$$\min \quad \frac{1}{2}\|\mathcal{C}y - z_d\|_{\mathcal{Z}}^2 + \frac{\gamma}{2}\|u\|_{\mathcal{U}}^2, \tag{2.1}$$

$$\text{s.t.} \quad a(y,v) + b(u,v) = l(v) \quad \forall v \in \mathcal{Y}. \tag{2.2}$$

Results on the existence of solutions for problems like (2.1), (2.2) are given e.g. in [1], [13] and we refer to those books.

We consider the following discretizations. Let

$$\mathcal{Y}_h = \text{span}\{\phi_1, \dots, \phi_{n_y}\} \subset \mathcal{Y}, \quad \mathcal{U}_h = \text{span}\{\psi_1, \dots, \psi_{n_u}\} \subset \mathcal{U},$$

and define matrices $A \in \mathbb{R}^{n_y \times n_y}$ and $B \in \mathbb{R}^{n_y \times n_u}$ by

$$\begin{aligned} A_{ij} &= a(\phi_j, \phi_i), \quad i,j = 1, \dots, n_y, \\ B_{ij} &= b(\psi_j, \phi_i), \quad j = 1, \dots, n_u, \ \ i = 1, \dots, n_y, \end{aligned}$$

and matrices $M_y \in \mathbb{R}^{n_y \times n_y}$ and $M_u \in \mathbb{R}^{n_u \times n_u}$ by

$$\begin{aligned} (M_y)_{ij} &= \langle \mathcal{C}\phi_j, \mathcal{C}\phi_i \rangle_{\mathcal{Z}}, \quad i,j = 1, \dots, n_y, \\ (M_u)_{ij} &= \langle \psi_j, \psi_i \rangle_{\mathcal{U}}, \quad i,j = 1, \dots, n_u. \end{aligned}$$

Obviously,

$$y_h^T M_y y_h = \|\sum_{i=1}^{n_y} y_{h,i}\mathcal{C}\phi_i\|_{\mathcal{Z}}^2, \quad u_h^T M_u u_h = \|\sum_{i=1}^{n_u} u_{h,i}\psi_i\|_{\mathcal{U}}^2.$$

In particular, the matrix M_u is positive definite and the matrix M_y is positive semidefinite.

By $\|\cdot\|$ we denote the Euclidean norm in $\mathbb{R}^k$ for some k.

We can show the following simple, but important result.

Lemma 2.1 *There exists a constant $c > 0$, independent of the discretization parameter h, such that*

$$\|M_y^{1/2} A^{-1} B M_u^{-1/2}\| \leq c.$$

Proof. Let $u_h \neq 0$ be arbitrary and set $y_h = A^{-1}BM_u^{-1/2}u_h$, $\tilde{u}_h = M_u^{-1/2}u_h$. Define $y = \sum_{i=1}^{n_y} y_{h,i}\phi_i$ and $\tilde{u} = \sum_{i=1}^{n_u} \tilde{u}_{h,i}\psi_i$. By definition of A and B, $a(y,\phi_i) = b(\tilde{u},\phi_i)$, $i = 1, \dots, n_y$. Hence,

$$\alpha\|y\|_{\mathcal{Y}}^2 \leq a(y,y) = b(\tilde{u},y) \leq \beta\|\tilde{u}\|_{\mathcal{U}}\|y\|_{\mathcal{Y}}.$$

This implies

$$\begin{aligned} \frac{\|M_y^{1/2}A^{-1}BM_u^{-1/2}u_h\|^2}{\|u_h\|^2} &= \frac{\|M_y^{1/2}y_h\|^2}{\|u_h\|^2} = \frac{\|\mathcal{C}y\|_{\mathcal{Z}}^2}{\|u_h\|^2} \\ &\leq \frac{\zeta^2\|y\|_{\mathcal{Y}}^2}{\|M_u^{1/2}\tilde{u}_h\|^2} = \frac{\zeta^2\|y\|_{\mathcal{Y}}^2}{\|\tilde{u}\|_{\mathcal{U}}^2} \leq \frac{\zeta^2\beta^2}{\alpha^2}. \end{aligned}$$

□

2.2 The Model Problem

The model problem (1.1)–(1.3) fits into the above framework, if we use the weak formulation of (1.2). The Hilbert spaces are $\mathcal{Y} = H^1(\Omega)$, $\mathcal{U} = L^2(\partial\Omega)$, and $\mathcal{Z} = L^2(\Omega)$. The bilinear forms and the functional are $a(y, v) = \int_\Omega \nabla y(x) \nabla v(x) + y(x)v(x)dx$, $b(u, v) = -\int_{\partial\Omega} u(x)v(x)dx$, and $l(v) = \int_\Omega f(x)v(x)dx$. The operator $\mathcal{C}$ is the imbedding operator. For our discretization we use a finite element discretization with piecewise linear functions over triangles. In our numerical experiments we use $\Omega = (0, 1)^2$ and we construct the triangulation as follows: The x– and y– intervals are subdivided into d_x and d_y subintervals. The resulting rectangles are subdivided into two triangles by connecting the lower left corner and the upper right corner of the rectangle. Since piecewise linear approximations are used, the number of state variables is $n_y = (d_x + 1)(d_y + 1)$ and the number of controls is $n_u = 2(d_x + d_y)$.

3 The Quadratic Programming Problem

We consider the following quadratic programming problem (QP) in standard form:

$$\text{Minimize } \frac{1}{2} \begin{pmatrix} y \\ u \end{pmatrix}^T \begin{pmatrix} M_{yy} & M_{yu} \\ M_{uy} & M_{uu} \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}^T \begin{pmatrix} y \\ u \end{pmatrix} \tag{3.1}$$

subject to

$$Ay + Bu = b, \tag{3.2}$$

$$y \geq 0, \ u \geq 0. \tag{3.3}$$

In this section the origin of the QP is not important and we omit the subscript h. Moreover, we absorb γ into M_{uu}. The standard form (3.1)–(3.3) is considered to reduce the complexity of notation. Using straightforward extensions, bound constraints of the form (1.6) can be handled as well. Throughout this section we use the notation

$$M = \begin{pmatrix} M_{yy} & M_{yu} \\ M_{uy} & M_{uu} \end{pmatrix}, \quad g = \begin{pmatrix} c \\ d \end{pmatrix}, \quad C = (A \mid B), \quad x = \begin{pmatrix} y \\ u \end{pmatrix}, \quad q = \begin{pmatrix} q_y \\ q_u \end{pmatrix}.$$

We limit our discussion to convex problems and assume that M is positive semidefinite. The existence of solutions of the QP (3.1)–(3.3) is guaranteed if the objective function is bounded from below on the set of feasible points. More precisely, we have the following well-known result (e.g. [6, § 12.3]):

Theorem 3.1 (Necessary and Sufficient Optimality Conditions) *If M is positive semidefinite and if $q(x) = \frac{1}{2}x^T Mx + g^T x$ is bounded from below on the set of feasible points $\{(y, u) | Ay + Bu = b, \ y \geq 0, u \geq 0\}$, then the QP (3.1)–(3.3) admits a solution x_*. If M is positive definite, the QP admits a unique solution.*

The vector (y, u) is a solution of (3.1)–(3.3) if and only if there exist $p \in \mathbb{R}^{n_y}$, $q_y \in \mathbb{R}^{n_y}$, and $q_u \in \mathbb{R}^{n_u}$ such that the Karush–Kuhn–Tucker (KKT) conditions

$$\begin{array}{rcl} M_{yy}y + M_{yu}u + A^T p - q_y &=& -d, \\ M_{uy}y + M_{uu}u + B^T p - q_u &=& -c, \\ Ay + Bu &=& b, \\ y^T q_y + u^T q_u &=& 0, \\ q_y, q_u &\geq& 0, \\ y, u &\geq& 0 \end{array} \tag{3.4}$$

are satisfied.

To learn more about the QP and the optimality system (3.4) it will be helpful to distinguish three cases. This discussion will also help us to relate the results in this paper to the results on the solution of KKT systems in interior–point methods for linear programming that can be found in the literature, see e.g. [10].

Throughout this subsection we assume that A is nonsingular and that the QP has a solution. As a consequence, the matrix $C = (A \mid B)$ has full row rank and the KKT system (3.4) has a solution.

Bound constraints for u and y. Let (y_*, u_*) be a solution of the QP. Furthermore, let $\{l_1^u, \dots, l_{k_u}^u\}$ and $\{l_1^y, \dots, l_{k_y}^y\}$ denote the set of active indices for u_* and y_*, respectively,

$$\{l_1^u, \dots, l_{k_u}^u\} = \{i \mid (u_*)_i = 0\}, \quad \{l_1^y, \dots, l_{k_y}^y\} = \{i \mid (y_*)_i = 0\}.$$

The Lagrange multipliers at the solution satisfy

$$(q_y)_i = 0,\ i \notin \{l_1^y, \dots, l_{k_y}^y\} \quad \text{and} \quad (q_u)_i = 0,\ i \notin \{l_1^u, \dots, l_{k_u}^u\}.$$

If we define the matrices $I(y_*) \in I\!R^{k_y \times n_y}$, $I(u_*) \in I\!R^{k_u \times n_u}$ by

$$(I(y_*))_{ij} = \begin{cases} 1 & \text{if } j = l_i^y, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad (I(u_*))_{ij} = \begin{cases} 1 & \text{if } j = l_i^u, \\ 0 & \text{otherwise,} \end{cases}$$

then the KKT conditions (3.4) are equivalent to

$$\begin{pmatrix} M_{yy} & M_{yu} & A^T & I(y_*)^T & 0 \\ M_{uy} & M_{uu} & B^T & 0 & I(u_*)^T \\ A & B & 0 & 0 & 0 \\ I(y_*) & 0 & 0 & 0 & 0 \\ 0 & I(u_*) & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \\ q_y^a \\ q_u^a \end{pmatrix} = \begin{pmatrix} -c \\ -d \\ b \\ 0 \\ 0 \end{pmatrix}, \tag{3.5}$$

where q_y^a, q_u^a denote the Lagrange multipliers corresponding to the active indices.

Let l denote the number of positive components in the solution (y_*, u_*) of the QP. The assumption that A is nonsingular is not sufficient to guarantee that the matrix

$$\widehat{C} = \begin{pmatrix} A & B \\ I(y_*) & 0 \\ 0 & I(u_*) \end{pmatrix} \in I\!R^{(n_y + (n_y + n_u - l)) \times (n_y + n_u)} \tag{3.6}$$

has full row rank. If $\widehat{C}$ does not have full rank, then the system (3.5) does not have a unique solution, even if the QP has a unique solution (y_*, u_*). It is not difficult to see that in this case the Lagrange multipliers (p, q_y^a, q_u^a) are not uniquely determined.

If $M = 0$, then the QP reduces to an LP. In this case the solution of the optimization problem can be found in a vertex (y_*, u_*). Recall that a feasible point (y, u) is called a vertex if the columns of $C = (A \mid B)$ corresponding to the positive components are linearly independent, see e.g. [6, § 2]. If (y_*, u_*) is a vertex, at most n_y components of (y_*, u_*) can be positive and the columns of $C = (A \mid B)$ corresponding to the positive components of the vertex (y_*, u_*) are linearly independent. If less than n_y components of (y_*, u_*) are positive the vertex is called *degenerate*, see e.g. [6, § 2]. In the nondegenerate case, i.e. if $l = n_y$ components of (y_*, u_*) are positive, then the matrix $\widehat{C}$ has full row rank. In the degenerate case, however, $l < n_y$ components of (y_*, u_*) are positive. Thus, $2n_y + n_u - l > n_y + n_u$ and the matrix $\widehat{C}$ cannot have full row rank. Hence, the solution is degenerate if and only if $\widehat{C}$ does not have full row rank.

Bound constraints for u. Let (y_*, u_*) be a solution of the QP and suppose that no bound constraints are imposed on y_* or that the bound constraints for y_* are not active. In this case,

$$\widehat{C} = \begin{pmatrix} A & B \\ 0 & I(u_*) \end{pmatrix}.$$

Since A is nonsingular, $\widehat{C}$ has full row rank. Therefore, the system (3.4) is uniquely solvable if the matrix M is positive definite on the null-space of $\overline{C}$.

In the LP case, i.e. $M = 0$, the solution can be found in a vertex (y_*, u_*). Since, by assumption, $y_* > 0$ and A is nonsingular, we can conclude that $u_* = 0$. Consequently, $I(u_*) \in I\!R^{n_u \times n_u}$ is the identity matrix. In the language of linear programming, y_* are the basis variables and u_* are the nonbasis variables. Thus, this case always corresponds to the *nondegenerate case* in linear programming.

No bound constraints. If the bound constraints are not active, then the Lagrange multipliers q_y and q_u are zero and the KKT conditions (3.4) are equivalent to the system (3.5) with the last two row and column blocks of the system matrix removed. If the matrix M is positive definite on the null–space of C, the system (3.4) has a unique solution.

4 Interior–Point Methods for the Solution of the Quadratic Programming Problem

It is not the purpose of this section to give an overview of interior point methods. We primarily address the structure of the linear systems arising in these methods to provide the necessary background for the construction of preconditioners. Because of space limitations, we focus on primal–dual interior–point methods. However, matrices with similar structure also arise in barrier methods, see e.g. [19] and [8], and certain affine–scaling methods, see e.g. [18].

We continue to use the notation of Section 3 and we will employ the notation common in interior point methods: For a given vector x, the diagonal matrix with diagonal entries equal to the entries of x is denoted by X. Moreover, e denotes the vector of ones, $e = (1, \ldots, 1)^T$.

The construction of primal–dual interior–point methods is based on the so–called perturbed KKT conditions corresponding to (3.4), which are given by

$$\begin{aligned} Mx + C^T p - q &= -g, \\ Cx &= b, \\ XQe &= \theta e, \end{aligned} \tag{4.7}$$

and $x, q > 0$, where $\theta > 0$. To move from a current iterate (x, p, q) with $x, q > 0$ to the next iterate (x_+, p_+, q_+), primal–dual Newton interior–point methods compute the Newton step $(\Delta x, \Delta p, \Delta q)$ for the perturbed KKT conditions (4.7) and set

$$(x_+, p_+, q_+) = (x + \alpha_x \Delta x, p + \alpha_p \Delta p, q + \alpha_q \Delta q),$$

where the step sizes $\alpha_x, \alpha_p, \alpha_q \in (0, 1]$ are chosen so that $x_+, q_+ > 0$. Then the perturbation parameter θ is updated based on $x_+^T q_+$ and the previous step is repeated. We refer to the literature, e.g. [20] for details.

The Newton system for the perturbed KKT conditions (4.7) is given by

$$\begin{pmatrix} M & C^T & -I \\ C & & \\ Q & & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta p \\ \Delta q \end{pmatrix} = - \begin{pmatrix} Mx + C^T p - q + g \\ Cx - b \\ XQe - \theta e \end{pmatrix}. \tag{4.8}$$

The nonsymmetric system (4.8) can be reduced to a symmetric system. If we use the last equation in (4.8) to eliminate Δq,

$$\Delta q = -X^{-1}Q\Delta x - Qe + \theta X^{-1}e, \tag{4.9}$$

then we arrive at the system

$$\begin{pmatrix} M + X^{-1}Q & C^T \\ C & \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta p \end{pmatrix} = - \begin{pmatrix} Mx + C^T p + g - \theta X^{-1}e \\ Cx - b \end{pmatrix}. \tag{4.10}$$

If $M_{yu} = 0, M_{uy} = 0$, the system (4.10) is of the form (1.7). As variables y_j or u_i approach the bound, i.e. approach zero, large quantities are added to the diagonals (j, j) or (i, i), respectively.

In actual computations more care must be taken during the reduction of the system (4.8) to avoid cancellation in the reduction process due to very large elements in X^{-1}, see e.g. [9]. A stable reduction of the system (4.8) is discussed in [9]. The unknowns and the right hand side in that reduced system differ from those in (4.10). However, the system matrix in the stable reduction is equal to the system matrix in (4.10). For our purposes it is therefore not necessary to present the lengthier stable reduction and we refer to [9] for details.

The influence of inexact solutions of the linear systems (4.10) onto the convergence behavior of the primal–dual interior–point method and the control of the inexactness is studied in [4], [12].

Before we continue, we briefly discuss the three cases explored in Section 3.

No bound constraints. In this case the diagonal contributions D_y and D_u coming from the interior–point method will be zero or close to zero. Since in our case the matrix M is positive definite, the system (3.4) has a unique solution. The ill-conditioning in the matrix K in this case is purely due to the discretization of the infinite dimensional control problem.

Bound constraints for u. It has been observed, e.g. [10], that in the nondegenerate case the KKT systems in barrier methods for linear programming can be preconditioned effectively. This will also be true in our case. If only bounds on u are active, efficient preconditioners can be constructed for the problems investigated in this paper. However, in our applications, ill–conditioning also arises from the matrices A. Although proven to be nonsingular, the matrices A arising in our applications have a wide spectrum which causes a large spread in the spectrum of the KKT matrix K. This will be investigated in more detail in Section 6.

Bound constraints for u **and** y. For the construction of preconditioners in barrier methods for linear programming the degenerate case is the difficult one. For example, the preconditioners discussed in [10] are far less effective in reducing the condition number of the KKT matrix in the degenerate case than they are in the nondegenerate case, cf. Tables 1 and 2 in [10]. This will also be the case in our situation. If bounds are only imposed on the controls u, efficient and rather general preconditioners can be derived. However, if state constraints, i.e. bounds on y, are present and active, then the QP (1.4)–(1.6) is very often degenerate and the design of preconditioners is much more difficult.

5 Solution of the Linear System

5.1 MINRES and SYMMLQ

Two Krylov subspace methods for the solution of indefinite linear systems, MINRES and SYMMLQ, have been introduced in [14]. These methods have been successfully used for problems like the one studied in this paper and are used for the solution of our systems.

We set $x = (y_h, u_h, p_h)^T$. Suppose the system to be solved is $Kx = b$. Given an initial iterate x_0 we set $r_0 = b - Kx_0$. The Krylov subspace $\mathcal{K}_j(K, r_0)$ is defined by

$$\mathcal{K}_j(K, r_0) = \text{span}\{r_0, Kr_0, \ldots, K^{j-1}r_0\}. \tag{5.1}$$

In iteration j, $j = 0, 1, \ldots$, the minimum residual method MINRES computes

$$x_j \in \mathcal{K}_j(K, r)$$

such that x_j solves

$$\min_{x \in \mathcal{K}_j(K, r_0)} \|r_0 - Kx\|.$$

In iteration j, $j = 0, 1, \ldots$, SYMMLQ computes the iterate

$$x_j \in \mathcal{K}_j(K, r_0)$$

such that x_j solves

$$(r_0 - Kx_j)^T v = 0 \quad \forall v \in \mathcal{K}_j(K, r_0).$$

Since K is indefinite, such an x_j may not exist. If it does not exist, SYMMLQ generates an iterate using information obtained from the Lanczos tridiagonalization. See [14].

The representation of Krylov subspaces (5.1) show that $x_j \in \mathcal{K}_j(K, r_0)$ if and only if

$$x_j = p_{j-1}(K)\, r_0,$$

where p_{j-1} is a polynomial of degree less or equal to $j-1$. This yields an upper bound for the residuals in MINRES:

$$\|r_0 - Kx_j\| = \|p_j^1(K) r_0\| \le \min_{p \in \Pi_j^1} \max_{\lambda \in \Lambda(K)} |p^1(\lambda)| \quad \|r_0\|. \tag{5.2}$$

Here $\Lambda(K)$ denotes the spectrum of K and Π_j^1 denotes the set of all polynomials p of degree less or equal to j which satisfy $p(0) = 1$. From (5.2) one can derive error estimates, see e.g. [16]. For example, using Chebyshev polynomials, one can show the following convergence estimate for MINRES:

$$\|r_0 - Kx_j\| \le 2 \left(\frac{\kappa - 1}{\kappa + 1} \right)^{\lfloor j/2 \rfloor} \|r_0\|,$$

where $\kappa = \bar{\lambda}/\underline{\lambda}$ is the condition number of K with $\underline{\lambda} = \min_{\lambda \in \Lambda(K)} |\lambda|$, $\bar{\lambda} = \max_{\lambda \in \Lambda(K)} |\lambda|$, and $\lfloor j/2 \rfloor$ is the largest integer less or equal to $j/2$.

If the matrix K has an unfavorable eigenvalue distribution, one constructs a nonsingular matrix P such that $K = P^{-1}KP^{-T}$ has a smaller condition number and better clustered eigenvalues. Instead of $Kx = b$ one solves the preconditioned system $\tilde{K}\tilde{x} = \tilde{b}$, where $\tilde{K} = P^{-1}KP^{-T}$, $\tilde{x} = P^T x$, and $\tilde{b} = P^{-1}b$. Of course, the preconditioner P has to be constructed so that matrix–vector multiplications with P^{-1} and P^{-T} can be done efficiently and so that the eigenvalue distribution of $P^{-1}KP^{-T}$ is improved.

For more details on MINRES and SYMMLQ we refer to [14], [2], and [3]. Those references also contain some details of the implementation. Complete listings of the preconditioned MINRES and SYMMLQ algorithms are given in [3]. We have implemented MINRES and SYMMLQ in Matlab.[1] Recently a version of the QMR algorithm has been developed in [9] to solve symmetric indefinite linear systems. These allow the application of indefinite preconditioners. If the preconditioner is positive definite, as in our case, then this QMR based method is equivalent to MINRES.

[1] A Fortran implementation of SYMMLQ written by M. Saunders is available from Netlib. See `linalg/symmlq` at `http://www.netlib.org/linalg/index.html`.

5.2 Eigenvalue Estimates

If A is invertible and if H_y and H_u are positive definite, then the matrix K defined by (1.7) has $n_y + n_u$ positive eigenvalues and n_y negative eigenvalues. More information on the eigenvalue distribution of K is provided by the following result, which is proven in [15]:

Lemma 5.1 (Rusten/Winther) *Suppose that H_y and H_u are positive definite and that $(A \mid B)$ has rank n_y. Let $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{n_y+n_u} > 0$ be the combined eigenvalues of H_y and H_u and let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n_y} > 0$ be the singular values of $(A \mid B)^T$. The eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{n_y+n_u} > 0 > \lambda_{n_y+n_u+1} \geq \ldots \geq \lambda_{2n_y+n_u}$ of K obey*

$$\lambda_{2n_y+n_u} \geq \frac{1}{2}(\mu_{n_y+n_u} - \sqrt{\mu^2_{n_y+n_u} + 4\sigma_1^2}), \tag{5.3}$$

$$\lambda_{n_y+n_u+1} \leq \frac{1}{2}(\mu_1 - \sqrt{\mu_1^2 + 4\sigma^2_{n_y}}), \tag{5.4}$$

$$\lambda_{n_y+n_u} \geq \mu_{n_y+n_u}, \tag{5.5}$$

$$\lambda_1 \leq \frac{1}{2}(\mu_1 + \sqrt{\mu_1^2 + 4\sigma_1^2}). \tag{5.6}$$

6 The Preconditioners

We now turn to the preconditioners for the matrix K in (1.7). We assume that $H_y \in I\!R^{n_y \times n_y}$, $H_u \in I\!R^{n_u \times n_u}$ are symmetric positive definite and that $A \in I\!R^{n_y \times n_y}$ is nonsingular.

In the following P_y and P_u are preconditioners of H_y and H_u, respectively, i.e. P_y and P_u are nonsingular matrices such that

$$P_y^{-1} H_y P_y^{-T} \approx I, \quad \text{and} \quad P_u^{-1} H_u P_u^{-T} \approx I. \tag{6.1}$$

By $\tilde{A}^{-1}$ we denote an approximate inverse of A,

$$\tilde{A}^{-1} A \approx I. \tag{6.2}$$

In our numerical tests we use $P_u = [\text{diag}(H_u)]^{1/2}$, $P_y = [\text{diag}(H_y)]^{1/2}$, and $\tilde{A} = A$. Since the diagonals of the mass matrices M_u and M_y are very good preconditioners for these matrices, these choices for the preconditioners P_u, P_y are efficient and satisfy (6.1).

In our computations we use K derived from the model problem and the finite element discretization outlined in Section 2.2. In all computations we use $d_x = d_y$. MINRES and SYMMLQ were used with starting value $x_0 = (y_h, u_h, p_h) = 0$ and the iterations were stopped when $\|P^{-1}b - P^{-1}KP^{-T}\tilde{x}_j\| < 10^{-5}$. We do not test our preconditioners within an interior–point method, but simulate the matrices K in (1.7) that would arise in an interior point method by adding diagonal matrices D_y and D_u. All computations are done in Matlab.

In the analysis of the preconditioners it will be helpful to distinguish four cases.

Case 1 ($\gamma = 1$, $D_y = 0$, $D_u = 0$): In this case we can reduce the condition number of the systems under consideration considerably. By preconditioning we reduce the iterations required by MINRES and SYMMLQ to a number which appears to be independent of the grid size.

Case 2 ($\gamma \ll 1$, $D_y = 0$, $D_u = 0$): In this case, the spectrum of H_u moves towards the origin, and while the conditioning of H_u itself is not changed, the condition number of K increases significantly. In this situation, ill–conditioning of K is induced by ill–posedness of the original problem. As γ decreases, the system with K becomes hard to solve, and for sufficiently small values of γ MINRES and SYMMLQ need an unacceptably large number of iterations. The performance of MINRES and SYMMLQ improves on the preconditioned systems.

Case 3 ($\gamma = 1$, $D_y = 0$, $D_u \gg I$): If bound constraints for u are active, corresponding diagonal entries in D_u increase. We write $D_u \gg I$ and mean this to be understood component wise. Large entries in D_u can be shown to affect the conditioning of the preconditioned system only to a moderate amount. In fact, they can even help to neutralize a small parameter γ or large entries in D_y. In this case our preconditioners are very effective.

Case 4 ($\gamma = 1$, $D_y \gg I$, $D_u = 0$): This case corresponds to the situation where bound constraints on y are active. As mentioned in Sections 3 and 4 the solution may be degenerate and this case may correspond to the degenerate case in linear programming. Often, a large diagonal in H_y unfavorably affects the performance of MINRES and SYMMLQ on the preconditioned systems. While the preconditioners introduced in the following lead to some improvement, their effectiveness in this case is much smaller than in the Cases 1 and 3. We point out that in our applications the number n_y of states is much larger than the number n_u of controls. Hence if more than n_u states are active at the solution, then the matrix $\widehat{C}$ in (3.6) can not have full row rank. In our numerical tests for Case 4 we set $D_y = 10^4 I$. This simulates the worst case in the sense that this corresponds to the case where all states approach the bounds. Our numerical tests always correspond to the degenerate case, which is the hard case.

6.1 The First Preconditioner

The first preconditioner is given by

$$P_1^{-1} = \begin{pmatrix} P_y^{-1} & 0 & 0 \\ 0 & P_u^{-1} & 0 \\ 0 & 0 & P_y^T \tilde{A}^{-1} \end{pmatrix}.$$

The preconditioned KKT matrix is

$$P_1^{-1} K P_1^{-T} = \begin{pmatrix} P_y^{-1} H_y P_y^{-T} & 0 & P_y^{-1} \tilde{A}^{-T} A^T P_y \\ 0 & P_u^{-1} H_u P_u^{-T} & P_u^{-1} B^T \tilde{A}^{-T} P_y \\ P_y^T \tilde{A}^{-1} A P_y^{-T} & P_y^T \tilde{A}^{-1} B P_u^{-T} & 0 \end{pmatrix} \tag{6.3}$$

and we expect that

$$P_1^{-1} K P_1^{-T} = \begin{pmatrix} \tilde{I}_{n_y} & 0 & \tilde{I}_{n_y} \\ 0 & \tilde{I}_{n_u} & P_u^{-1} B^T \tilde{A}^{-T} P_y \\ \tilde{I}_{n_y} & P_y^T \tilde{A}^{-1} B P_u^{-T} & 0 \end{pmatrix}, \tag{6.4}$$

where $\tilde{I}$ is an approximate identity matrix. The preconditioned system still has the structure allowing us to estimate its spectrum using Lemma 5.1. The derivation of the general form of our first preconditioner is motivated by the assumption that for preconditioners P_y, P_u of H_y, H_u and for an approximate inverse $\tilde{A}^{-1}$ of A the singular values of

$$\tilde{B} = P_y^T \tilde{A}^{-1} B P_u^{-T} \tag{6.5}$$

are of moderate size. If $P_y = M_y^{1/2}$, $P_u = M_u^{1/2}$, and $\tilde{A} = A$, this is guaranteed in the situation of Section 2.1. See Lemma 2.1.

Lemma 6.1 *Let $\tilde{B} \in I\!R^{n_y \times n_u}$. The singular values σ_i of $(I_m|\tilde{B})$ are given by*

$$\sigma_i = \sqrt{1 + \sigma_i^2(\tilde{B})}, \quad i = 1, \ldots, n_y,$$

where $\sigma_i(\tilde{B})$ are the singular values of $\tilde{B}$. If $n_y \geq n_u$, $\tilde{B}$ has n_u singular values, and we set $\sigma_i(\tilde{B}) = 0$ for $i = n_u + 1, \ldots, n_y$.

Proof. The proof follows immediately from the fact that the squares of the singular values of a matrix B are the eigenvalues of BB^T. $\square$

In the situation of Section 2.1 the estimate in Lemma 2.1 shows that

$$\sigma_i(\tilde{B}) \leq \|M_y^{1/2} A^{-1} B M_u^{-1/2}\| \leq c, \quad i = 1, \ldots, n_y, \tag{6.6}$$

for a constant c independent of h. Thus in Case 1 ($H_y = M_y$ and $H_u = M_u$) we expect that, for preconditioners P_u, P_y and $\tilde{A}$ neutralizing the dependency of H_y, H_u and A on the mesh constant h, we can similarly bound the singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$ such that

$$\sigma_i(\tilde{B}) \leq \|P_y^T \tilde{A}^{-1} B P_u^{-T}\| \leq c_P, \tag{6.7}$$

where c_P is a constant independent of h.

Assuming that (6.7) is valid we discuss the expected performance of the first preconditioner in the four cases defined earlier. By $\sigma_i^{(l)} = \sigma_i^{(l)}(P_y^T \tilde{A}^{-1} B P_u^{-T})$, $l = 1, 2, 3, 4$, we denote the singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$ in Case $l = 1, 2, 3, 4$.

Case 1 ($\gamma = 1$, $D_y = 0$, $D_u = 0$): If $\gamma = 1$, (6.7) shows that there exists a constant upper bound for the singular values $\sigma^{(1)}(H_y^{1/2} A^{-1} B H_u^{-1/2})$. The preconditioner P_1 can be expected to perform well if the preconditioning matrices P_y, P_u and $\tilde{A}$ neutralize the influence of the mesh size h on the submatrices and thus on the system, and if the singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$ are bounded by a small constant c_P. If the eigenvalues of $P_y^{-1} H_y P_y^{-1}$ and $P_u^{-T} H_u P_u^{-1}$ are close to one and if $\sigma_{min}^{(1)} \ll 1$, where $\sigma_i^{(1)}$ denote the singular values of $(P_y^T \tilde{A}^{-1} B P_u^{-T})$, we can deduce

$$\lambda_{n_y + n_u} \approx 1, \quad \lambda_{n_y + n_u + 1} \approx \frac{1}{2}(1 - \sqrt{5}),$$

so that the eigenvalues of the preconditioned system are bounded away from zero. If in addition $\sigma_{max}^{(1)}$, i.e. the constant c_P in (6.7) is of moderate size, Lemma 5.1 guarantees that the condition number of the preconditioned system $P_1^{-1} K P_1^{-T}$ is small. MINRES and SYMMLQ will perform very well on the preconditioned system. This is confirmed by our numerical tests. See Table 1.

The preconditioner will perform poorly if the singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$ are not small. This happens in two of the remaining three cases.

Case 2 ($\gamma \ll 1$, $D_y = 0$, $D_u = 0$): If a small parameter γ determines the size of the eigenvalues of the matrix M_u, we must expect that bounds on the norm $\|H_y^{1/2} A^{-1} B H_u^{-1/2}\|$ grow with the reciprocal of $\sqrt{\gamma}$. Denoting by $\sigma_i^{(2)}$ the singular values of $H_y^{1/2} A^{-1} B H_u^{-1/2}$, we have the relationship

$$\sigma_i^{(2)} = \frac{1}{\sqrt{\gamma}} \sigma_i^{(1)}.$$

For decreasing values of γ the spectrum of $P_y^T \tilde{A}^{-1} B P_u^{-T}$ expands and the conditioning of the preconditioned system deteriorates.

Case 3 ($\gamma = 1$, $D_y = 0$, $D_u \gg I$): In this case $H_u = \gamma M_u + D_u$, where $D_u \gg I$, i.e. some diagonal entries may become very large. Analogously we write $P_u = \gamma P_O + P_D$, where P_D stands for the (large) diagonal entries and P_O for the off–diagonal entries that are generally of moderate size. By $\sigma_i^{(3)}$ we denote the singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$. We obtain the estimate

$$\begin{aligned}
\sigma_i^{(3)} &= \sigma_i^{(3)}(P_y^T \tilde{A}^{-1} B P_u^{-T}) \\
&= \sigma_i^{(3)}(P_y^T \tilde{A}^{-1} B (\gamma P_O + P_D)^{-T}) \\
&= \sigma_i^{(3)}(P_y^T \tilde{A}^{-1} B P_D^{-T} (\gamma P_D^{-1} P_O + I)^{-T}) \\
&\leq \|P_y^T \tilde{A}^{-1} B\| \, \|P_D^{-T}\| \, \|\gamma (P_D^{-1} P_O + I)^{-T}\| \\
&\leq \|P_y^T \tilde{A}^{-1} B\| \, \|P_D^{-T}\| \, \frac{1}{1 - \|\gamma P_O^T P_D^{-T}\|}.
\end{aligned}$$

If D_u dominates the matrix H_u, $\|\gamma P_O P_D^{-T}\|$ will be of negligible size. If additionally $\gamma \ll 1$, this contributes to reducing the factor $1/(1 - \|\gamma P_O P_D^{-1}\|)$ to a constant close to one. The norm $\|P_y^T \tilde{A}^{-1} B\|$ can be expected to be of moderate size, while $\|P_D^{-1}\|$ will be very small. The singular values $\sigma^{(3)}$ converge to zero as the entries in the diagonal D_u, and with it in P_D, grow. In the case of large diagonal entries in H_u we can expect a good performance of the solvers on the preconditioned system, due to a small condition number of $P_1^{-1} K P_1^{-T}$ which is in turn induced by small singular values of $P_y^T \tilde{A}^{-1} B P_u^{-T}$. The performance of MINRES and SYMMLQ on the preconditioned system is documented in Table 2.

Case 4 ($\gamma = 1$, $D_y \gg I$, $D_u = 0$): If we denote by P_y the preconditioner for H_y and by P_O, P_D its off-diagonal part and its diagonal part, respectively, then we see that the matrix $P_y^T \tilde{A}^{-1} B P_u^{-T}$ will have very large singular values. This is indicated by the estimates ($M = \tilde{A}^{-1} B P_u^{-T} P_u^{-1} B^T \tilde{A}^{-T}$)

$$\lambda_{max}((P_O + P_D)^T M (P_O + P_D)) \geq \lambda_{max}(P_D^T M P_D) + \lambda_{min}(P_O^T M P_O + P_O^T M P_D + P_D^T M P_O)$$

and

$$\lambda_{min}((P_O + P_D)^T M (P_O + P_D)) \leq \lambda_{min}(P_O^T M P_D + P_D^T M P_O + P_D^T M P_D) + \lambda_{max}(P_O^T M P_O).$$

For the estimates see [11, p. 411]. While the preconditioner yields a considerable improvement over the unpreconditioned system, the improvement is less than in Cases 1 and 3. See Table 3. However, the improvement is expected to decrease as the diagonals in D_y become larger.

6.2 The Second Preconditioner

We have seen that the effectiveness of preconditioner P_1 depends on the size of the singular values of the matrix $\tilde{B}$ defined in (6.5). The preconditioner P_2 is designed to isolate the effect of $\tilde{B}$. In order to make the action of the second preconditioner transparent, we consider the ideal version of P_2, denoted by P_2^*, i.e. we choose $P_u = H_u^{1/2}$, $P_y = H_y^{1/2}$, and $\tilde{A} = A$. For the general form of the preconditioner, which is used in the computations, we refer to [3].

The ideal preconditioner P_2^* is given by its inverse as

$$(P_2^*)^{-1} = \begin{pmatrix} H_y^{-1/2} & 0 & 0 \\ 0 & H_u^{-1/2} & 0 \\ -H_y^{-1/2} & -H_y^{1/2} A^{-1} B H_u^{-1} & H_y^{1/2} A^{-1} \end{pmatrix}.$$

The ideal preconditioned system is

$$P_2^{*-1} K P_2^{*-T} = \begin{pmatrix} I_{n_y} & 0 & 0 \\ 0 & I_{n_u} & 0 \\ 0 & 0 & -(I_{n_y} + \tilde{B}\tilde{B}^T) \end{pmatrix},$$

where $\tilde{B}$ is defined by (6.5) with $P_u = H_u^{1/2}$, $P_y = H_y^{1/2}$, and $\tilde{A} = A$.

The application of the preconditioner P_2 is roughly as expensive as the application of the preconditioner P_1. The performance of P_2 is slightly inferior to the performance of P_1. See Tables 1–3. The eigenvalue distribution of the preconditioned system, i.e. the eigenvalue distribution of $(I_{n_y} + \tilde{B}\tilde{B}^T)$, can be analyzed analogously to the previous case.

6.3 The Third Preconditioner

A third preconditioner is derived from reductions performed to solve QP subproblems in sequential quadratic programming methods, see e.g. [7]. As before we use the ideal form for the presentation of the preconditioner. The general form of the preconditioner, see [3], is used in the computations. The ideal preconditioner P_3^*, given by its inverse as

$$(P_3^*)^{-1} = \begin{pmatrix} I_{n_y} & 0 & -1/2\, H_y A^{-1} \\ 0 & 0 & A^{-1} \\ -(A^{-1}B)^T & I_{n_u} & (A^{-1}B)^T\, H_y A^{-1} \end{pmatrix},$$

transforms K into the preconditioned system

$$(P_3^*)^{-1} K (P_3^*)^{-T} = \begin{pmatrix} 0 & I_{n_y} & 0 \\ I_{n_y} & 0 & 0 \\ 0 & 0 & W^T H W \end{pmatrix},$$

where

$$W = \begin{pmatrix} -A^{-1}B \\ I_{n_u} \end{pmatrix}, \quad H = \begin{pmatrix} H_y & 0 \\ 0 & H_u \end{pmatrix}.$$

The matrix W is a representation for the nullspace of $C = (A|B)$. The matrix $W^T H W$ is given by

$$W^T H W = B^T A^{-T} H_y A^{-1} B + H_u = H_u^{1/2}\left(\tilde{B}^T\tilde{B} + I_{n_u}\right) H_u^{1/2},$$

where $\tilde{B}$ is defined by (6.5) with $P_u = H_u^{1/2}$, $P_y = H_y^{1/2}$, and $\tilde{A} = A$. Note that the partitioning of the blocks in the preconditioned system has changed.

The preconditioner P_3 is the most effective in reducing the number of iterations. See Tables 1–3. However, the application of the general preconditioner P_3 is roughly twice as expensive as the application of the preconditioners P_1 and P_2. See [3]. The eigenvalue distribution of $W^T H W$ can be analyzed analogously to the preconditioned system with P_1.

Table 1
Iterations of MINRES and SYMMLQ on K with $\gamma = 1$, $D_y = 0$, $D_u = 0$.

grid size d_x	5	10	15	20	25	30
dimension	92	282	572	962	1452	2042
Without Preconditioning						
MINRES	47	185	431	784	1070	1483
SYMMLQ	47	179	407	647	902	1209
Preconditioner P_1						
MINRES	23	25	24	21	21	19
SYMMLQ	23	24	22	21	19	19
Preconditioner P_2						
MINRES	24	35	37	37	35	35
SYMMLQ	24	35	36	35	35	33
Preconditioner P_3						
MINRES	7	6	5	5	5	4
SYMMLQ	7	6	5	5	5	4

7 Conclusions

In this paper we have derived preconditioners for matrices K arising in the numerical solution of certain distributed linear quadratic control problems by interior–point methods. The preconditioners are in block form, with blocks composed of preconditioners for the individual blocks of the matrix K. This allows the incorporation of known preconditioners for the governing equations of the original problem and it allows to reuse computationally expensive information within all interior–point iterations. The effectiveness of the preconditioners was analyzed using the properties of the control problem and its discretization, the block structure of the matrix K, and information from the optimality conditions. Numerical results supporting the theoretical analysis were given.

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Table 2
Iterations of MINRES and SYMMLQ for K with $\gamma = 1$ and $D_u = 10^4 \cdot I$, $D_y = 0$.

grid size d_x	5	10	15	20	25	30
dimension	92	282	572	962	1452	2042
Without Preconditioning						
MINRES	54	173	349	589	857	1183
SYMMLQ	54	173	349	579	848	1165
Preconditioner P_1						
MINRES	16	18	18	18	18	16
SYMMLQ	16	18	18	18	17	16
Preconditioner P_2						
MINRES	21	33	35	37	35	35
SYMMLQ	21	33	35	35	33	33
Preconditioner P_3						
MINRES	5	4	4	4	4	4
SYMMLQ	5	4	4	4	4	4

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Table 3
Iterations of MINRES and SYMMLQ for K with $\gamma = 1$ and $D_y = 10^4 \cdot I$, $D_u = 0$.

grid size d_x	5	10	15	20	25	30
dimension	92	282	572	962	1452	2042
Without Preconditioning						
MINRES	73	282	572	962	1452	2042
SYMMLQ	73	282	572	962	1452	2042
Preconditioner P_1						
MINRES	50	98	194	289	449	530
SYMMLQ	50	98	187	283	410	524
Preconditioner P_2						
MINRES	61	146	235	323	453	583
SYMMLQ	61	143	233	323	447	547
Preconditioner P_3						
MINRES	44	67	120	203	275	366
SYMMLQ	44	56	120	167	286	355

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