Degree-bounded vertex partitions

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Abstract

This paper studies degree-bounded vertex partitions, considers analogues for well-known results on the chromatic number and graph perfection, and presents two algorithms for constructing degree-bounded vertex partitions. The first algorithm minimizes the number of partition classes. The second algorithm minimizes a weighted sum of the partition classes where the weight of a partition class depends on the level of adjacency among its vertices.

1 Introduction

A *coloring* partitions the vertex set of a graph $G = (V, E)$ into subsets of pairwise non-adjacent vertices. A classical problem in combinatorial optimization is to find a coloring which uses the smallest possible number of color classes. The minimum number of color classes required is known as the chromatic number $\chi(G)$. If V represents a set of objects and E the set of conflicting pairs, graph coloring solves the problem of dividing V into the minimum number of conflict-free subgroups.

A second application of graph coloring arises from its relation to another classical problem in combinatorial optimization. The maximum clique problem asks for the largest subset of pairwise adjacent vertices in a graph. Since the color classes of a coloring are edgeless, a subset of pairwise adjacent vertices meets each color class at most once. Consequently, $\chi(G)$ is an upper bound on the cardinality of a maximum clique, and researchers [2, 21, 23] use graph coloring in branch and bound solvers for the maximum clique problem.

Generalized graph coloring describes the partitioning of the vertices into classes whose induced subgraphs satisfy particular constraints [22]. For example, k-improper colorings have the property that each color class induces a subgraph of maximum degree at most k [1, 7, 12]. The generalization to k-improper colorings suggests two optimization problems.

The first seeks to partition a graph into degree-bounded induced subgraphs using the smallest possible number of partition classes. This solves the problem of dividing V into the minimum number of subgraphs such that each vertex has a bounded number of conflicts in its partition class. The number of partition classes required defines the k-improper chromatic number and has been studied in a variety of contexts [6, 11, 13]. Much of this research focuses on random graphs and generalizations of the Four Color Theorem. Some applications of k-improper coloring include radio-frequency assignment [11] and network security [19].

The second problem minimizes a weighted sum of the partition classes. In this case, the weight of a partition class depends on the level of adjacency among its vertices. This second problem produces a bound on the cardinality of subgraphs defined as degree-based clique relaxations [18] and leads to a generalization of graph perfection. This optimization problem appears to be new, and the present study is motivated by its role as an upper bound for clique relaxations.

The remainder of this paper is organized as follows. Section 2 discusses some relevant definitions and notation. Section 3 explores the relationship between degree-based clique relaxations and degree-bounded vertex partitions. Section 4.1 adapts a well-known graph coloring algorithm [15] to solve the problem of minimizing the number of partition sets. Section 4.2 uses the algorithm from Section 4.1 to solve a more general class of problems. Section 5 summarizes and suggests some future research directions.

2 Preliminaries

All graphs $G = (V, E)$ in this paper are finite, undirected, and simple. The *girth*, $g(G)$, is the length of the smallest cycle in G. Given a vertex $v \in V$, define $N_G(v) := \{u \in V \mid uv \in E\}$, $deg_G(v) := |N_G(v)|$, $\Delta(G) := max_{v \in V}\ deg_G(v)$, and $\delta(G) := min_{v \in V}\ deg_G(v)$. Let $G[K]$ denote the subgraph induced by $K \subseteq V$. In this paper, $k \geq 1$ is always a positive integer.

Definition 1. $K \subseteq V$ induces a *k-plex* if $\delta(G[K]) \geq |K| - k$.

Definition 2. $C \subseteq V$ induces a *co-k-plex* if $\Delta(G[C]) \leq k - 1$.

Definition 3. A partition of the vertex set into disjoint, nonempty co-k-plexes defines a *co-k-plex coloring* of G.

Definitions 1 and 2 are due to Seidman and Foster [20]. Co-k-plexes are also known as $(k-1)$-dependent or $(k-1)$-stable sets [11]. Notice that 1-plexes and co-1-plexes are complete subgraphs and stable sets, respectively. Let $\omega_k(G)$ denote the cardinality of a largest k-plex in G, $\alpha_k(G)$ the cardinality of a largest co-k-plex in G, and Π the set of all co-k-plex colorings of G.

Definition 4. The *co-k-plex chromatic number* of G is defined as

$$\chi_k(G) := min\left\{\sum_{C \in P} \omega_k(G[C]) \ : \ P \in \Pi\right\}.$$

Definition 5. The *cardinality co-k-plex chromatic number* of G is defined as

$$\bar{\chi}_k(G) := min\{m \ : \ \exists P \in \Pi \ s.t.\ |P| = m\}.$$

$\bar{\chi}_k(G)$ is exactly the $(k-1)$-improper chromatic number [22]. A $\bar{\chi}_k$*-optimal* coloring partitions V using the smallest possible number of co-k-plex sets. A χ_k*-optimal* coloring

$C_1, ..., C_m$ satisfies $\chi_k(G) = \sum_{i=1}^{m} \omega_k(G[C_i])$ and thus

$$\bar{\chi}_k(G) \leq m \leq \sum_{i=1}^{m} \omega_k(G[C_i]) = \chi_k(G).$$

Notice also that $\chi_1(G) = \bar{\chi}_1(G) = \chi(G)$. Moreover, a coloring is χ_1-optimal if and only if it is $\bar{\chi}_1$-optimal. However, this relationship fails for $k > 1$. To see this, consider the trivial example of k pairwise non-adjacent vertices. The unique $\bar{\chi}_k$-optimal coloring consists of a single color class. On the other hand, assigning each vertex to a distinct color class defines a χ_k-optimal coloring which uses k color classes.

A co-k-plex C is called *deficient* whenever $|C| < k$. A deficient co-k-plex C satisfies $\omega_k(G[C]) = |C|$. A *compact* co-k-plex coloring has at most one deficient co-k-plex set.

Lemma 1. *Every co-k-plex coloring $C_1, ..., C_m$ can be changed into a compact co-k-plex coloring $C'_1, ..., C'_p$ such that $p \leq m$ and $\sum_{i=1}^{m} \omega_k(G[C_i]) = \sum_{i=1}^{p} \omega_k(G[C'_i])$.*

Proof. Consider the co-k-plex coloring $C_1, ..., C_m$. Suppose there are two deficient co-k-plexes C_i and C_j. It follows that $\omega_k(G[C_i]) + \omega_k(G[C_j]) = |C_i| + |C_j|$. Choose a vertex $v \in C_j$. Define $C'_j := C_j \setminus \{v\}$ and $C'_i := C_i \cup \{v\}$. Now $|C_i \cup \{v\}| \leq k$ ensures that C'_i and C'_j both remain co-k-plexes. Moreover,

$$\omega_k(G[C'_i]) + \omega_k(G[C'_j]) = (|C_i| + 1) + (|C_j| - 1) = \omega_k(G[C_i]) + \omega_k(G[C_j]).$$

Continue moving vertices from C_j to C_i until either $C'_j = \varnothing$ or $|C'_i| = k$, in which case the number of deficient sets has been reduced. This procedure can be repeated until the co-k-plex coloring $C'_1, ..., C'_p$ is compact. It is also clear that $p \leq m$ since the procedure can only reduce the number of partition sets in the co-k-plex coloring. □

3 Bounding degree-based clique relaxations

This section analyzes the relationship between $\chi_k(G)$ and $\omega_k(G)$. Section 3.1 introduces the notion of k-plex perfection, offers some examples of k-plex perfect graphs, and explores k-plex analogues for certain properties of perfection. Section 3.2 discusses the gap between $\chi_k(G)$ and $\omega_k(G)$.

3.1 k-plex perfection

A coloring function partitions V into co-1-plexes to obtain an upper bound on $\omega_1(G)$. Similarly, partitioning V into degree-bounded subgraphs leads to an upper bound on $\omega_k(G)$. Let $S_1, ..., S_m$ be a co-k-plex coloring of G, and let $K \subseteq V$ be a maximum k-plex in G. Observe that

$$\omega_k(G) = |K| = \sum_{i=1}^{m} |K \cap S_i| \leq \sum_{i=1}^{m} \omega_k(G[S_i]), \tag{1}$$

where the inequality follows from the fact that k-plexes are closed under set inclusion [20]. Notice that $\chi_k(G) \geq \omega_k(G)$. Recall that a graph G is perfect if $\chi(G') = \omega_1(G')$ for every vertex-induced subgraph $G' \subseteq G$.

Definition 6. A *k-plex perfect* graph G satisfies $\omega_k(G') = \chi_k(G')$ for all vertex-induced subgraphs $G' \subseteq G$.

For example, a co-k-plex S satisfies $\chi_k(S) = \omega_k(S)$ by definition. Therefore, co-k-plexes are k-plex perfect because every vertex-induced subgraph of a co-k-plex is also a co-k-plex [20]. Recall that a finite set X and a family $\mathcal{I}$ of subsets of X define a *matroid* if the following axioms hold:

1. $\varnothing \in \mathcal{I}$
2. $I' \subseteq I \in \mathcal{I}$ implies $I' \in \mathcal{I}$
3. Every maximal set in $\mathcal{I}$ has the same cardinality

Given a graph $G = (V, E)$, define

$$\mathcal{K} = \{K \subseteq V \ : \ \delta(G[K]) \geq |K| - k\}.$$

$\mathcal{K}$ is the set of k-plexes in G, and $(V, \mathcal{K})$ satisfies the first two matroid axioms for any graph.

Theorem 1. *If $M := (V, \mathcal{K})$ defines a matroid, then G is k-plex perfect.*

Proof. Given any vertex-induced subgraph $G' = (V', E')$, define $D := V \setminus V'$ and $\mathcal{K}' = \{K \subseteq V' \ : \ \delta(G[K]) \geq |K| - k\}$. Observe that

$$(V', \mathcal{K}') = (V \setminus D, \mathcal{K}') =: M \setminus D$$

is again a matroid known as a deletion matroid, so it suffices to show $\chi_k(G) = \omega_k(G)$.

Define $x(A) = \sum_{a \in A} x_a$, $\mathcal{S} = \{S \subseteq V \ : \ \Delta(G[S]) \leq k - 1\}$, and $\mathcal{S}_v = \{S \in \mathcal{S} \ : \ v \in S\}$. Consider the following dual pair of linear programs:

$$max\{x(V) \ : \ x \geq 0, \ x(S) \leq \omega_k(G[S]) \text{ for all } S \in \mathcal{S}\} \tag{2}$$

$$min\{\sum_{S \in \mathcal{S}} \omega_k(G[S]) y_S \ : \ y \geq 0, \ y(\mathcal{S}_v) \geq 1 \text{ for all } v \in V\}. \tag{3}$$

Since M is a matroid, a theorem of Edmonds [9] implies that optimal solutions for (2) and (3) are integral. Observe that $\omega_k(G)$ and $\chi_k(G)$ are the optimal objective values for (2) and (3), respectively. Moreover, $\omega_k(G) = \chi_k(G)$ by strong duality. □

Corollary 1. *If G is a k-plex, then G is k-plex perfect.*

Proof. Given any $K' \subset V$ and $v \in V \setminus K'$, $K' \cup \{v\}$ defines a k-plex. It follows that all maximal k-plexes have cardinality $\omega_k(G) = |V|$, so G is k-plex perfect by Theorem 1. □

Recall that an r-partite graph is r-colorable. The complete r-partite graphs have all possible edges between distinct color classes.

Theorem 2. *If G is the complete r-partite graph $K_{n_1,\ldots,n_r}$, then G is k-plex perfect.*

Proof. The proof will show that all maximal k-plexes in G have the same cardinality. The result then follows from Theorem 1. Let K be a maximal k-plex in G and S_i the i^{th} partition class. Clearly, $|K \cap S_i| \leq |S_i| = n_i$. In addition, $|K \cap S_i| \leq k$. For if not, let $v \in K \cap S_i$, and notice that $N_G(v) \cap S_i = \varnothing$ implies

$$deg_{G[K]}(v) = |K| - |K \cap S_i| < |K| - k,$$

which contradicts that K is a k-plex. Therefore, $|K \cap S_i| \leq min\{k, n_i\}$ for each S_i.

Suppose for contradiction that $|K| = \sum_{i=1}^{r} |K \cap S_i| < \sum_{i=1}^{r} min\{k, n_i\}$. Then there exists a j such that $|K \cap S_j| < min\{k, n_j\}$, and $|K \cap S_j| < n_j$ implies that there exists a vertex $v \in S_j \setminus K$. Consider the set $K' := K \cup \{v\}$ and a vertex $u \in K' \setminus S_j$. Since $uv \in E$,

$$deg_{G[K']}(u) = deg_{G[K]}(u) + 1 \geq (|K| - k) + 1 = |K'| - k.$$

Now suppose $u \in K \cap S_j$. Observe that $deg_{G[K']}(u) = deg_{G[K]}(u) = |K| - |K \cap S_j| > |K| - k$ since $uv \notin E$ and $|K \cap S_j| < k$. It follows that

$$deg_{G[K']}(u) \geq |K| - k + 1 = |K'| - k.$$

Thus, since $deg_{G[K']}(u) = deg_{G[K']}(v)$, K' is a k-plex in G, which contradicts the maximality of K. It follows that all maximal k-plexes in G have cardinality $\sum_{i=1}^{r} min\{k, n_i\}$, so G is k-plex perfect by Theorem 1. □

It turns out that many properties of perfect graphs do not have k-plex analogues. Consider the complement $\overline{K}_{r,r}$ of a complete bipartite graph. Both components H_1 and H_2 of $\overline{K}_{r,r}$ are complete subgraphs.

Lemma 2. *Let $k \geq 1$. If $r = 2k - 1$, then $\alpha_k(\overline{K}_{r,r}) = 2k$ and $\omega_k(\overline{K}_{r,r}) = 2k - 1$.*

Proof. In the proof of Theorem 2, it was shown that

$$\omega_k(K_{r,r}) = \sum_{i=1}^{2} min\{k, r\} = 2k.$$

Thus, $\alpha_k(\overline{K}_{r,r}) = \omega_k(K_{r,r}) = 2k$.

Now $\omega_k(\overline{K}_{r,r}) \geq 2k-1$ because each component H_i is complete and hence a k-plex of cardinality $2k-1$. Suppose for contradiction that $\omega_k(\overline{K}_{r,r}) > 2k-1$. Then there exists a k-plex $K \subseteq V$ such that $|K| = 2k$. If $|K \cap H_i| \leq k$, then

$$deg_{\overline{K}_{r,r}[K]}(v) \leq k-1 < k = |K| - k \text{ for all } v \in K \cap H_i.$$

This contradicts the definition of k-plex. Therefore, $|K \cap H_1| > k$ and $|K \cap H_2| > k$, which contradicts $|K| = 2k$. □

Theorem 3. *Let $k > 1$. If $r = 2k-1$, then $\overline{K}_{r,r}$ is not k-plex perfect.*

Proof. By Lemma 2, it suffices to show that $\chi_k(\overline{K}_{r,r}) \geq 2k$. Clearly, $\chi_k(\overline{K}_{r,r}) \geq \omega_k(\overline{K}_{r,r}) = 2k-1$. Suppose for contradiction that $\chi_k(\overline{K}_{r,r}) = 2k-1$. Lemma 1 implies the existence of a χ_k-optimal coloring $S_1, ..., S_m$ of $\overline{K}_{r,r}$ such that $|S_1| \geq k$. Therefore, $\omega_k(\overline{K}_{r,r}[S_1]) \geq k$. Furthermore, $\chi_k(\overline{K}_{r,r}) < 2k$ implies that all other sets S_i satisfy $|S_i| < k$. Notice that

$$2k-1 = \chi_k(\overline{K}_{r,r}) = \sum_{i=1}^{m} \omega_k(\overline{K}_{r,r}[S_i]) \geq k + \sum_{i=2}^{m} \omega_k(\overline{K}_{r,r}[S_i]) = k + \sum_{i=2}^{m} |S_i|.$$

Consequently, $k-1 \geq \sum_{i=2}^{m} |S_i|$. Now since the sets S_i partition V and $|V| = 4k-2$,

$$|S_1| = |V| - \sum_{i=2}^{m} |S_i| \geq 3k-1.$$

Therefore, $k > 1$ implies that $|S_1| \geq 3k-1 > 2k$. This contradicts Lemma 2 because S_1 is a co-k-plex and $\alpha_k(\overline{K}_{r,r}) = 2k$. □

Lovász's [16] replication lemma is a well-known result from the theory of perfect graphs.

Replication of a vertex $v \in V$ corresponds to the following operation: create a new vertex v' and join it to v and all the neighbors of v. The replication lemma states that replication of a vertex in a perfect graph produces another perfect graph. However, for $k \geq 2$, replication of a vertex in a k-plex perfect graph does not necessarily produce another k-plex perfect graph.

Fix $k > 1$. Consider the edgeless graph G on two vertices v_1 and v_2. G is a co-k-plex since $\Delta(G) = 0$. It follows that G is k-plex perfect. Construct G' by performing $2k - 2$ replication operations on each of v_1 and v_2. This procedure creates $G' = \overline{K}_{r,r}$, which is not k-plex perfect by Theorem 3. Therefore, vertex replication does not preserve k-plex perfection. Theorem 3 also illustrates the following interesting property: G might not be k-plex perfect even if all components of G are k-plex perfect. This statement follows from Corollary 1 and Theorem 3.

The final topic of this section is a k-plex version of the Weak Perfect Graph Theorem [16]. The Weak Perfect Graph Theorem states that G is perfect if and only if $\overline{G}$ is perfect. Theorems 2 and 3 together provide counterexamples for k-plex analogues of the Weak Perfect Graph Theorem for any $k \geq 2$.

3.2 The duality gap

In 1959, Erdős [10] showed that the difference $\chi_1(G) - \omega_1(G)$ can be arbitrarily large. More precisely, he showed that for every integer $r \geq 1$, there exists a graph G' with girth $g(G') > r$ and chromatic number $\chi(G') > r$. Observe that $g(G) > 3$ implies $\omega_1(G) \leq 2$. Therefore, the theorem establishes the existence of graphs with large chromatic number and small clique number. Analogously, one might ask if the gap between $\chi_k(G)$ and $\omega_k(G)$ can also become arbitrarily large.

Lemma 3. *For every graph G, $\chi(G) \leq \chi_k(G)$.*

Proof. Let $C_1, ..., C_m$ be a χ_k-optimal coloring of G. The proof is by induction on m. If

$m = 1$, then G is a co-k-plex and

$$\chi(G) \leq \Delta(G) + 1 \leq \min\{|V| - 1, k - 1\} + 1 = \min\{|V|, k\} \leq \omega_k(G) \leq \chi_k(G).$$

The first inequality is well-known [8]. The second and third inequalities follow from the definitions of co-k-plex and k-plex, respectively. Section 3.1 derives the last inequality (1).

If $m > 1$, then

$$\chi(G) \leq \chi(G - C_1) + \chi(G[C_1]) \leq \chi_k(G - C_1) + \chi_k(G[C_1]) = \chi_k(G).$$

The second inequality follows from induction. The equality follows since C_1 is part of a χ_k-optimal coloring. $\square$

Lemma 3 and the Erdős theorem together imply that for every integer $r \geq 1$, there exists a graph G' with $g(G') > r$ and $\chi_k(G') > r$.

Corollary 2. *Given any integer $r > k + 2$, there exists a graph G with $\chi_k(G) > r$ and $\omega_k(G) < k + 2$.*

Proof. If $\omega_k(G) \geq k + 2$, then G contains a k-plex K of cardinality $k + 2$. Moreover, $\delta(G[K]) \geq 2$ by definition of k-plex. It follows that $G[K] \subseteq G$ contains a cycle of length at most $k + 2 = |K|$. Therefore, $g(G) > k + 2$ implies that $\omega_k(G) < k + 2$. $\square$

4 Algorithms

This section develops algorithms for finding degree-bounded vertex partitions. Section 4.1 contains an exact $\bar{\chi}_k$-coloring algorithm. Section 4.2 shows how to find χ_2-optimal colorings using the $\bar{\chi}_2$-coloring algorithm. Sections 4.1 and 4.2 both contain computational results. All implementations were run on a 2.2 GHz Dual-Core AMD Opteron processor with 3 GB of memory.

```
function implicitENUM(G, n)
    ub = n + 1; r = 1
    loop
        FORWARDS(r)
        BACKWARDS(r)
        if r = 0 then break
    repeat
end

function FORWARDS(r)
    for i = r to n
        reorder uncolored vertices v_i, ..., v_n
        if r = 1 or r < i then determine FC(i)
        if FC(i) = ∅ then r = i; return
        C'(i) = min(FC(i))
    repeat
    C = C'; ub = max(C)
    r = least i such that C(i) = ub
end

function BACKWARDS(r)
    CP = {1, ..., r − 1}
    while CP ≠ ∅
        i = max(CP); CP = CP − {i}
        FC(i) = FC(i) − {C'(i)}
        if FC(i) ≠ ∅ then r = i; return
    repeat
    r = 0
end
```

Figure 1: A generalized implicit enumeration algorithm [15].

4.1 $\bar{\chi}_k$-optimal coloring

In [15], Kubale and Jackowski present a generalized implicit enumeration algorithm for graph coloring which subsumes a number of previous combinatorial approaches [3, 4, 5, 14]. This section adapts the Kubale and Jackowski algorithm to find $\bar{\chi}_k$-optimal colorings.

Figure 4.1 contains the generalized implicit enumeration algorithm as given in [15]. Before running the algorithm, the vertex set of G is ordered $(v_1, ..., v_n)$ such that $deg_G(v_i) \geq deg_G(v_{i+1})$. The vertex ordering can either remain static or change dynamically throughout the algorithm. The array C' stores a partial co-k-plex coloring. The array C stores the incumbent co-k-plex coloring. For each $1 \leq i \leq n$, $FC(i)$ stores the set of feasible colors for v_i with respect to the current partial coloring C'. In other words, $FC(i)$ consists of partition classes S such that $S \cup \{v_i\}$ is a co-k-plex. CP is the set of current predecessors. These vertices are the candidates for backtracking.

The main difference between traditional graph coloring and $\bar{\chi}_k$-coloring is the structure of the partition classes, so adapting the coloring algorithm in Figure 4.1 amounts to finding an appropriate definition for the set of feasible colors $FC(i)$. Given the partial co-k-plex coloring $S_1, ..., S_r$, define

$$P(i) = \{j : S_j \cup \{v_i\} \text{ is not a co-}k\text{-plex}\} \cup \{\ ub\}.$$

The set of feasible colors is defined as $FC(i) = \{1, 2, ..., max_{j<i}C'(j) + 1\} - P(i)$. This definition forces each partition class to be a co-k-plex.

For $k = 2, 3, 4$, the algorithm was tested on a set of random graphs and a subset of the DIMACS coloring instances. The random graph GN-P has N vertices and edge probability $\frac{P}{100}$. Table 1 contains a description of the test instances.

Tables 2 and 3 contain computational results obtained by running two versions of this algorithm. Each table reports the runtime in seconds and the number of branch and bound nodes (BBN). BBN counts the number of loops performed in **implicitENUM**. In the first version, $A1$, the vertex ordering remains static. For the second version, $A2$, the vertex ordering is dynamic. In determining the order, the algorithm always colors the vertex v_i such that $|P(i)|$ is maximum. Ties are broken by choosing the vertex of larger degree. This dynamic reordering is analogous to DSATUR [3].

Both algorithms appear to perform better on sparse graphs. $A2$ dominates $A1$ on all instances except for the graph *jean*. This suggests that it is worthwhile to use a dynamic reordering scheme.

Table 1: Test instances

G	$\lvert V\rvert$	$\lvert E\rvert$	G	$\lvert V\rvert$	$\lvert E\rvert$	G	$\lvert V\rvert$	$\lvert E\rvert$
G20-10	20	21	G60-50	60	902	jean*	80	508
G20-30	20	56	G60-70	60	1244	myciel3*	11	20
G20-50	20	98	G60-90	60	1587	myciel4*	23	71
G20-70	20	127	G80-10	80	311	myciel5*	47	236
G20-90	20	170	G80-30	80	919	myciel6*	95	755
G40-10	40	65	G80-50	80	1583	myciel7*	191	2360
G40-30	40	223	G80-70	80	2191	queen5-5*	25	320
G40-50	40	379	G80-90	80	2876	queen6-6*	36	580
G40-70	40	539	anna*	138	986	queen7-7*	49	952
G40-90	40	709	david*	87	812	queen8-12*	96	2736
G60-10	60	173	games120*	120	1276	queen8-8*	64	1456
G60-30	60	546	homer*	561	3258	queen9-9*	81	2112

* DIMACS graph

4.2 χ_2-optimal coloring

Section 4.1 shows how traditional graph coloring algorithms can solve the $\bar{\chi}_k$-coloring problem. However, these algorithms do not apply directly to χ_k-coloring since a χ_k-coloring algorithm must consider the weight $\omega_k(G[S_i])$ of each partition class in a co-k-plex coloring $S_1, ..., S_m$.

This section focuses on the χ_2-coloring problem. The proposed algorithm effectively reduces χ_2-coloring to $\bar{\chi}_2$-coloring. Lemma 1 implies that the set of compact co-k-plex colorings always contains a χ_k-optimal coloring. As a result, an algorithm can restrict the search for a χ_k-optimal solution by considering only compact co-k-plex colorings. For $k = 2$, Lemma 1 has an even stronger consequence. Recall that $\omega_2(G[C]) = min\{2, |C|\} \in \{1, 2\}$ for any nonempty co-2-plex C [17].

Let Π_{min} be the set of compact $\bar{\chi}_2$-optimal colorings. Define $\tilde{C}_1, ..., \tilde{C}_m$ as follows: If possible, choose a co-2-plex coloring in Π_{min} with a deficient set, and let $\tilde{C}_m$ be that deficient set. If no such co-2-plex coloring exists, choose an arbitrary co-2-plex coloring in Π_{min}.

Lemma 4. *$\tilde{C}_1, ..., \tilde{C}_m$ is a χ_2-optimal coloring.*

Proof. Suppose for contradiction that $\tilde{C}_1, ..., \tilde{C}_m$ is not χ_2-optimal. $\tilde{C}_1, ..., \tilde{C}_m$ is compact, so $|\tilde{C}_i| \geq 2$ for $i = 1, ..., m-1$. Therefore, $\sum_{i=1}^{m} \omega_2(G[\tilde{C}_i]) = 2(m-1) + min\{2, |\tilde{C}_m|\}$. Now consider a χ_2-optimal coloring $C_1, ..., C_r$. By Lemma 1, assume that $C_1, ..., C_r$ is compact.

Table 2: $\bar{\chi}_k$-coloring Algorithm $A1$

G	$\bar{\chi}_2(G)$	sec.	BBN	$\bar{\chi}_3(G)$	sec.	BBN	$\bar{\chi}_4(G)$	sec.	BBN
G20-10	2	0.00	5	2	0.00	4	2	0.00	6
G20-30	3	0.00	15	3	0.00	22	2	0.00	13
G20-50	4	0.00	32	3	0.00	552	3	0.00	99
G20-70	6	0.00	1300	4	0.00	647	4	0.05	7339
G20-90	7	0.13	17509	5	0.30	38043	5	23.96	2765462
G40-10	3	0.00	122	2	0.00	5	2	0.00	9
G40-30	4	0.03	3338	4	0.16	18399	3	0.00	401
G40-50	6	1.38	129385	5	9.80	1030246	4	4.33	390503
G40-70	9	1871.99	170648924	7	≥18000	1617576331	6	≥18000	1452584759
G40-90	14	≥18000	1211734838	10	≥18000	1248208098	8	≥18000	1211346362
G60-10	3	0.00	12	3	0.00	526	2	0.04	2875
G60-30	6	3244.19	340703615	5	13226.1	1220520832	4	177.91	18335449
G60-50	9	≥18000	1251395534	7	≥18000	1122692457	7	≥18000	1004914660
G60-70	13	≥18000	1003359489	11	≥18000	894198034	10	≥18000	845373017
G60-90	20	≥18000	774059901	17	≥18000	678163929	12	≥18000	757167324
G80-10	3	62.96	4140070	3	0.03	3271	3	68.83	4184865
G80-30	7	≥18000	993233487	6	≥18000	807014223	5	≥18000	1067096687
G80-50	12	≥18000	770055557	11	≥18000	645035483	9	≥18000	600895813
G80-70	17	≥18000	665399344	14	≥18000	626925720	12	≥18000	608747780
G80-90	31	≥18000	459055967	22	≥18000	508500361	17	≥18000	522780218
anna	6	0.08	8525	4	6.23	320375	4	0.05	7613
david	7	≥18000	1384045099	5	1.30	156810	5	≥18000	1448152514
games120	6	≥18000	723520664	5	≥18000	836731644	3	3062.61	91452239
homer	9	≥18000	795490717	6	≥18000	738855316	5	1480.93	137771886
jean	5	3.08	488763	4	14.08	1803128	3	7.13	803339
myciel3	2	0.00	4	2	0.00	6	2	0.00	8
myciel4	3	0.00	32	2	0.00	7	2	0.00	8
myciel5	4	0.22	24409	3	0.01	608	3	0.04	4888
myciel6	5	≥18000	1280433317	4	431.05	33505972	3	60.82	4282158
myciel7	8	≥18000	578439709	5	≥18000	821907790	6	≥18000	426881829
queen5-5	5	0.53	56195	4	0.03	3393	3	0.05	4841
queen6-6	6	1.67	159863	4	33.29	3099994	4	1.89	170639
queen7-7	6	8342.97	587654870	5	7944.05	508997845	4	1656.75	107454753
queen8-12	11	≥18000	575538363	8	≥18000	784867186	7	≥18000	752565847
queen8-8	8	≥18000	935902497	6	≥18000	1169352037	5	≥18000	1138121220
queen9-9	9	≥18000	763674243	7	≥18000	845499313	7	≥18000	691545778

* upper bound

Table 3: $\bar{\chi}_k$-coloring Algorithm $A2$

G	$\bar{\chi}_2(G)$	sec.	BBN	$\bar{\chi}_3(G)$	sec.	BBN	$\bar{\chi}_4(G)$	sec.	BBN
G20-10	2	0.00	3	2	0.00	3	2	0.00	5
G20-30	3	0.00	4	3	0.00	15	2	0.00	4
G20-50	4	0.00	16	3	0.00	224	3	0.00	40
G20-70	6	0.02	952	4	0.01	499	4	0.14	5910
G20-90	7	0.36	15575	5	0.68	30893	5	52.20	2628085
G40-10	3	0.00	7	2	0.00	3	2	0.00	6
G40-30	4	0.00	45	4	0.05	687	3	0.00	51
G40-50	6	0.14	1688	5	3.05	36617	4	1.75	20637
G40-70	9	230.35	2850720	7	≥18000	218968533	6	≥18000	229358981
G40-90	14	≥18000	260234647	10	≥18000	271052619	8	≥18000	303224783
G60-10	3	0.00	4	3	0.00	37	2	0.00	9
G60-30	6	42.67	212012	5	1142.11	6267784	4	69.14	366802
G60-50	8	2370.52	11440627	7	≥18000	93538078	6	≥18000	95346333
G60-70	12	≥18000	91150112	11	≥18000	91577723	10	≥18000	110104109
G60-90	20	≥18000	126526427	17	≥18000	162496400	12	≥18000	136157091
G80-10	3	0.33	1078	3	0.01	63	3	1.82	6717
G80-30	7	≥18000	55714765	6	≥18000	53859728	5	≥18000	57760372
G80-50	11	≥18000	43993617	9	≥18000	54157311	8	≥18000	49116474
G80-70	16	≥18000	50835631	14	≥18000	55777148	12	≥18000	59458530
G80-90	30	≥18000	77234043	22	≥18000	91024606	17	≥18000	95975487
anna	6	0.26	1058	4	0.48	2183	4	1.00	5977
david	6	0.17	1331	5	16.97	132507	4	0.62	6343
games120	5	≥18000	82627193	4	≥18000	60957615	3	0.14	606
homer	9	≥18000	10554232	6	≥18000	12507768	5	10242.1	9826931
jean	6	≥18000	198779251	5	≥18000	227798062	4	≥18000	266816752
myciel3	2	0.00	2	2	0.00	4	2	0.00	7
myciel4	3	0.00	9	2	0.00	11	2	0.00	14
myciel5	4	0.04	413	3	0.00	85	3	0.02	254
myciel6	5	1739.92	5517741	4	648.84	1997558	4	≥18000	49595236
myciel7	6	≥18000	19972596	5	≥18000	15556934	5	≥18000	15874666
queen5-5	5	0.08	1838	4	0.01	461	3	0.02	416
queen6-6	6	0.73	8916	4	15.71	194542	4	1.89	25140
queen7-7	6	614.61	4045323	5	562.06	3911896	4	327.35	2296973
queen8-12	9	≥18000	39419276	8	≥18000	38194398	7	≥18000	42744690
queen8-8	7	3303.87	14182717	6	≥18000	82842458	5	≥18000	86059606
queen9-9	9	≥18000	54145884	7	≥18000	52867510	6	≥18000	54373060

* upper bound

Therefore, $\chi_2(G) = \sum_{i=1}^{r} \omega_2(G[C_i]) = 2(r-1) + min\{2, |C_r|\}$, where C_r is the deficient set if one exists in $C_1, ..., C_r$. Finally, observe that

$$2(m-1) + min\{2, |\tilde{C}_m|\} > \chi_2(G) = 2(r-1) + min\{2, |C_r|\},$$

which implies

$$min\{2, |\tilde{C}_m|\} - min\{2, |C_r|\} > 2(r-m).$$

Case 1: Suppose $r = m$. It follows that $min\{2, |\tilde{C}_m|\} - min\{2, |C_r|\} > 0$. Thus $min\{2, |\tilde{C}_m|\} = 2$ and $min\{2, |C_r|\} = 1$. In other words, $\tilde{C}_m$ is not deficient; C_r is deficient; and $C_1, ..., C_r$ belongs to Π_{min}. This contradicts the choice of $\tilde{C}_1, ..., \tilde{C}_m$.

Case 2: Suppose $r > m$. Then $min\{2, |\tilde{C}_m|\} - min\{2, |C_r|\} > 2$, a contradiction since $min\{2, |\tilde{C}_m|\} \in \{1, 2\}$ and $min\{2, |C_r|\} \in \{1, 2\}$.

□

Lemma 4 reduces the problem of finding a χ_2-optimal coloring to that of finding an element in Π_{min}. This is desirable as the algorithm from Section 4.1 can solve the latter problem. The proposed algorithm consists of two steps, both of which make use of implicit enumeration.

The first step constructs a compact $\bar{\chi}_2$-optimal coloring. The second step searches for a compact $\bar{\chi}_2$-optimal coloring with one deficient set. If such a co-2-plex coloring exists, it is χ_2-optimal by Lemma 4. If no such co-2-plex coloring exists, then the co-2-plex coloring from the first step is χ_2-optimal by Lemma 4.

The first step uses the coloring algorithm exactly as described in Section 4.1. In the second step, for each $v \in V$, the algorithm uses implicit enumeration to search for a $\bar{\chi}_2(G)-1$ coloring of $G - v$. If step two manages to find such a coloring in $G - v$, then v is a deficient co-2-plex in a $\bar{\chi}_2$-optimal coloring of G. $A2$ was used for the implicit enumeration. Table 4 contains computational results obtained by running the algorithm on the instances which were solved to $\bar{\chi}_2$-optimality by $A2$.

Table 4: χ_2-coloring Algorithm

G	$\chi_2(G)$	sec.	BBN	G	$\chi_2(G)$	sec.	BBN
G20-10	4	0.00	23	G60-50	16	2522.57	12612916
G20-30	6	0.00	24	G80-10	6	0.33	1158
G20-50	8	0.00	76	anna	12	2.63	15548
G20-70	12	0.06	3052	david	12	1.04	10504
G20-90	14	0.77	34475	myciel3	4	0.00	13
G40-10	5	0.00	7	myciel4	6	0.00	32
G40-30	8	0.00	165	myciel5	8	0.06	554
G40-50	12	0.36	5958	myciel6	10	1804.36	5541140
G40-70	18	602.75	8514132	queen5-5	10	0.09	2217
G60-10	6	0.00	64	queen6-6	12	1.00	14163
G60-30	12	44.12	227480	queen7-7	12	616.69	4052767

Notice that most of these graphs satisfy $\chi_2(G) = 2 \cdot \bar{\chi}_2(G)$. These are exactly the graphs where step two of the algorithm failed to find a coloring with a deficient co-2-plex. However, the algorithm did find such a coloring for the graph $G40$-10.

5 Conclusion

This paper studies degree-bounded vertex partitions. Section 3 studies χ_k-optimal colorings, defines k-plex perfection, and offers examples of k-plex perfect graphs. It is also shown that many properties of graph perfection do not have k-plex analogues. Section 3.2 analyzes the gap between $\chi_k(G)$ and $\omega_k(G)$.

This paper also derives algorithms for constructing degree-bounded vertex partitions. Section 4.1 offers a straightforward generalization of a traditional graph coloring algorithm. The resulting algorithm partitions the vertex set into the minimum number of co-k-plexes. Section 4.2 shows how to find χ_2-optimal colorings by reducing the problem to that of finding $\bar{\chi}_2$-optimal colorings. Lemma 4 represents a key step in the reduction. An interesting open problem is to determine if $\bar{\chi}_k$-coloring can be used to solve χ_k-coloring problems for $k > 2$. Another avenue for future research is a polyhedral analysis of the co-k-plex coloring polytope.

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