

Degree-bounded vertex partitions

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Abstract

This paper studies degree-bounded vertex partitions, derives analogues for well-known results on the chromatic number and graph perfection, and presents two algorithms for constructing degree-bounded vertex partitions. The first algorithm minimizes the number of partition classes. The second algorithm minimizes a weighted sum of the partition classes where the weight of a partition class depends on the level of adjacency among its vertices.

1 Introduction

A *coloring* partitions the vertex set of a graph $G = (V, E)$ into subsets of pairwise non-adjacent vertices. A classical problem in combinatorial optimization is to find a coloring which uses the smallest possible number of color classes. The minimum number of color classes required is known as the chromatic number $\chi(G)$. If V represents a set of objects and E the set of conflicting pairs, graph coloring solves the problem of dividing V into the minimum number of conflict-free subgroups.

A second application of graph coloring arises from its relation to another classical problem in combinatorial optimization. The maximum clique problem asks for the largest subset of pairwise adjacent vertices in a graph. Since the color classes of a coloring are edgeless, a subset of pairwise adjacent vertices meets each color class at most once. Consequently, $\chi(G)$ is an upper bound on the cardinality of a maximum clique, and researchers [2, 21, 23] use graph coloring in branch and bound solvers for the maximum clique problem.

Generalized graph coloring describes the partitioning of the vertices into classes whose induced subgraphs satisfy particular constraints [22]. For example, k -improper colorings have the property that each color class induces a subgraph of maximum degree at most k [1, 7, 12]. The generalization to k -improper colorings suggests two optimization problems.

The first seeks to partition a graph into degree-bounded subgraphs using the smallest possible number of partition classes. This solves the problem of dividing V into the minimum number of subgroups such that each vertex has a bounded number of conflicts in its partition class. The second problem minimizes a weighted sum of the partition classes. In this case, the weight of a partition class depends on the level of adjacency among its vertices. This second problem produces a bound on the cardinality of subgraphs defined as degree-based clique relaxations [18] and leads to a generalization of graph perfection.

The k -improper chromatic number is associated with the first optimization problem and has been studied in a variety of contexts [6, 11, 13]. Much of this research focuses on random graphs and generalizations of the Four Color Theorem. Some applications of k -improper coloring include radio-frequency assignment [11] and network security [19]. The second optimization problem appears to be new.

The remainder of this paper is organized as follows. Section 2 discusses some relevant definitions and notation. Section 3 explores the relationship between cohesive subgraphs and degree-bounded vertex partitions. Section 4.1 adapts a well-known graph coloring algorithm [15] to solve the problem of minimizing the number of partition sets. Section 4.2 uses the algorithm from Section 4.1 to solve a more general class of problems. Section 5 summarizes and suggests some future research directions.

2 Preliminaries

All graphs $G = (V, E)$ in this paper are finite, undirected, and simple. The *girth*, $g(G)$, is the length of the smallest cycle in G . Given a vertex $v \in V$, define $N_G(v) := \{u \in V \mid uv \in E\}$,

$deg_G(v) := |N_G(v)|$, $\Delta(G) := \max_{v \in V} deg_G(v)$, and $\delta(G) := \min_{v \in V} deg_G(v)$. Let $G[K]$ denote the subgraph induced by $K \subseteq V$. In this paper, $k \geq 1$ is always a positive integer.

Definition 1. $K \subseteq V$ induces a k -plex if $\delta(G[K]) \geq |K| - k$.

Definition 2. $C \subseteq V$ induces a co - k -plex if $\Delta(G[C]) \leq k - 1$.

Definition 3. A partition of the vertex set into disjoint, nonempty co - k -plexes defines a co - k -plex coloring of G .

Definitions 1 and 2 are due to Seidman and Foster [20]. Co - k -plexes are also known as $(k - 1)$ -dependent or $(k - 1)$ -stable sets [11]. Notice that 1-plexes and co -1-plexes are complete subgraphs and stable sets, respectively. Let $\omega_k(G)$ denote the cardinality of a largest k -plex in G , $\alpha_k(G)$ the cardinality of a largest co - k -plex in G , and Π the set of all co - k -plex colorings of G .

Definition 4. The co - k -plex chromatic number of G is defined as

$$\chi_k(G) := \min \left\{ \sum_{C \in P} \omega_k(G[C]) : P \in \Pi \right\}.$$

Definition 5. The cardinality co - k -plex chromatic number of G is defined as

$$\bar{\chi}_k(G) := \min \{ m : \exists P \in \Pi \text{ s.t. } |P| = m \}.$$

$\bar{\chi}_k(G)$ is exactly the $(k - 1)$ -improper chromatic number [22]. A $\bar{\chi}_k$ -optimal coloring partitions V using the smallest possible number of co - k -plex sets. A χ_k -optimal coloring C_1, \dots, C_m satisfies $\chi_k(G) = \sum_{i=1}^m \omega_k(G[C_i])$ and thus

$$\bar{\chi}_k(G) \leq m \leq \sum_{i=1}^m \omega_k(G[C_i]) = \chi_k(G).$$

Notice also that $\chi_1(G) = \bar{\chi}_1(G) = \chi(G)$. Moreover, a coloring is χ_1 -optimal if and only if it is $\bar{\chi}_1$ -optimal. However, this relationship fails for $k > 1$. To see this, consider the trivial

example of k pairwise non-adjacent vertices. The unique $\bar{\chi}_k$ -optimal coloring consists of a single color class. On the other hand, assigning each vertex to a distinct color class defines a χ_k -optimal coloring which uses k color classes.

A co- k -plex C is called *deficient* whenever $|C| < k$. A deficient co- k -plex C satisfies $\omega_k(G[C]) = |C|$. A *compact* co- k -plex coloring has at most one deficient co- k -plex set.

Lemma 1. *Every co- k -plex coloring C_1, \dots, C_m can be changed into a compact co- k -plex coloring C'_1, \dots, C'_p such that $p \leq m$ and $\sum_{i=1}^m \omega_k(G[C_i]) = \sum_{i=1}^p \omega_k(G[C'_i])$.*

Proof. Consider the co- k -plex coloring C_1, \dots, C_m . Suppose there are two deficient co- k -plexes C_i and C_j . It follows that $\omega_k(G[C_i]) + \omega_k(G[C_j]) = |C_i| + |C_j|$. Choose a vertex $v \in C_j$. Define $C'_j := C_j \setminus \{v\}$ and $C'_i := C_i \cup \{v\}$. Now $|C_i \cup \{v\}| \leq k$ ensures that C'_i and C'_j both remain co- k -plexes. Moreover,

$$\omega_k(G[C'_i]) + \omega_k(G[C'_j]) = (|C_i| + 1) + (|C_j| - 1) = \omega_k(G[C_i]) + \omega_k(G[C_j]).$$

Continue moving vertices from C_j to C_i until either $C'_j = \emptyset$ or $|C'_i| = k$, in which case the number of deficient sets has been reduced. This procedure can be repeated until the co- k -plex coloring C'_1, \dots, C'_p is compact. It is also clear that $p \leq m$ since the procedure can only reduce the number of partition sets in the co- k -plex coloring. \square

3 Bounding cohesive subgraphs

This section analyzes the relationship between $\chi_k(G)$ and $\omega_k(G)$. Section 3.1 introduces the notion of k -plex perfection, offers some examples of k -plex perfect graphs, and explores k -plex analogues for certain properties of perfection. Section 3.2 discusses a theorem of Erdős [10].

3.1 k -plex perfection

A coloring function partitions V into co-1-plexes to obtain an upper bound on $\omega_1(G)$. Similarly, partitioning V into degree-bounded subgraphs leads to an upper bound on $\omega_k(G)$. Let S_1, \dots, S_m be a co- k -plex coloring of G , and let $K \subseteq V$ be a maximum k -plex in G . Observe that

$$\omega_k(G) = |K| = \sum_{i=1}^m |K \cap S_i| \leq \sum_{i=1}^m \omega_k(G[S_i]),$$

where the inequality follows from the fact that k -plexes are closed under set inclusion [20]. Notice that $\chi_k(G) \geq \omega_k(G)$. Recall that a graph G is perfect if $\chi(G') = \omega_1(G')$ for every vertex-induced subgraph $G' \subseteq G$.

Definition 6. A k -plex perfect graph G satisfies $\omega_k(G') = \chi_k(G')$ for all vertex-induced subgraphs $G' \subseteq G$.

For example, a co- k -plex S satisfies $\chi_k(S) = \omega_k(S)$ by definition. Therefore, co- k -plexes are k -plex perfect because every vertex-induced subgraph of a co- k -plex is also a co- k -plex [20]. Recall that a finite set X and a family \mathcal{I} of subsets of X define a *matroid* if the following axioms hold:

1. $\emptyset \in \mathcal{I}$
2. $I' \subseteq I \in \mathcal{I}$ implies $I' \in \mathcal{I}$
3. Every maximal set in \mathcal{I} has the same cardinality

Given a graph $G = (V, E)$, define

$$\mathcal{K} = \{K \subseteq V : \delta(G[K]) \geq |K| - k\}.$$

\mathcal{K} is the set of k -plexes in G , and (V, \mathcal{K}) satisfies the first two matroid axioms for any graph.

Theorem 1. *If $M := (V, \mathcal{K})$ defines a matroid, then G is k -plex perfect.*

Proof. Given any vertex-induced subgraph $G' = (V', E')$, define $D := V \setminus V'$ and $\mathcal{K}' = \{K \subseteq V' : \delta(G[K]) \geq |K| - k\}$. Observe that

$$(V', \mathcal{K}') = (V \setminus D, \mathcal{K}') =: M \setminus D$$

is again a matroid known as a deletion matroid, so it suffices to show $\chi_k(G) = \omega_k(G)$.

Define $x(A) = \sum_{a \in A} x_a$, $\mathcal{S} = \{S \subseteq V : \Delta(G[S]) \leq k - 1\}$, and $\mathcal{S}_v = \{S \in \mathcal{S} : v \in S\}$.

Consider the following dual pair of linear programs:

$$\max\{x(V) : x \geq 0, x(S) \leq \omega_k(G[S]) \text{ for all } S \in \mathcal{S}\} \quad (1)$$

$$\min\left\{\sum_{S \in \mathcal{S}} \omega_k(G[S])y_S : y \geq 0, y(\mathcal{S}_v) \geq 1 \text{ for all } v \in V\right\}. \quad (2)$$

Since M is a matroid, a theorem of Edmonds [9] implies that optimal solutions for (1) and (2) are integral. Observe that $\omega_k(G)$ and $\chi_k(G)$ are the optimal objective values for (1) and (2), respectively. Moreover, $\omega_k(G) = \chi_k(G)$ by strong duality. \square

Corollary 1. *If G is a k -plex, then G is k -plex perfect.*

Proof. Given any $K' \subset V$ and $v \in V \setminus K'$, $K' \cup \{v\}$ defines a k -plex. It follows that all maximal k -plexes have cardinality $\omega_k(G) = |V|$, so G is k -plex perfect by Theorem 1. \square

Recall that an r -partite graph is r -colorable. The complete r -partite graphs have all possible edges between distinct color classes.

Theorem 2. *If G is the complete r -partite graph K_{n_1, \dots, n_r} , then G is k -plex perfect.*

Proof. The proof will show that all maximal k -plexes in G have the same cardinality. The result then follows from Theorem 1. Let K be a maximal k -plex in G and S_i the i^{th} partition class. Clearly, $|K \cap S_i| \leq |S_i| = n_i$. In addition, $|K \cap S_i| \leq k$. For if not, let $v \in K \cap S_i$, and

notice that $N_G(v) \cap S_i = \emptyset$ implies

$$\deg_{G[K]}(v) = |K| - |K \cap S_i| < |K| - k,$$

which contradicts that K is a k -plex. Therefore, $|K \cap S_i| \leq \min\{k, n_i\}$ for each S_i .

Suppose for contradiction that $|K| = \sum_{i=1}^r |K \cap S_i| < \sum_{i=1}^r \min\{k, n_i\}$. Then there exists a j such that $|K \cap S_j| < \min\{k, n_j\}$, and $|K \cap S_j| < n_j$ implies that there exists a vertex $v \in S_j \setminus K$. Consider the set $K' := K \cup \{v\}$ and a vertex $u \in K' \setminus S_j$. Since $uv \in E$,

$$\deg_{G[K']}(u) = \deg_{G[K]}(u) + 1 \geq (|K| - k) + 1 = |K'| - k.$$

Now suppose $u \in K \cap S_j$. Observe that $\deg_{G[K']}(u) = \deg_{G[K]}(u) = |K| - |K \cap S_j| > |K| - k$ since $uv \notin E$ and $|K \cap S_j| < k$. It follows that

$$\deg_{G[K']}(u) \geq |K| - k + 1 = |K'| - k.$$

Thus, since $\deg_{G[K']}(u) = \deg_{G[K']}(v)$, K' is a k -plex in G , which contradicts the maximality of K . It follows that all maximal k -plexes in G have cardinality $\sum_{i=1}^r \min\{k, n_i\}$, so G is k -plex perfect by Theorem 1. \square

It turns out that many properties of perfect graphs do not have k -plex analogues. Consider the complement $\overline{K}_{r,r}$ of a complete bipartite graph. Both components H_1 and H_2 of $\overline{K}_{r,r}$ are complete subgraphs.

Lemma 2. *Let $k \geq 1$. If $r = 2k - 1$, then $\alpha_k(\overline{K}_{r,r}) = 2k$ and $\omega_k(\overline{K}_{r,r}) = 2k - 1$.*

Proof. In the proof of Theorem 2, it was shown that

$$\omega_k(K_{r,r}) = \sum_{i=1}^2 \min\{k, r\} = 2k.$$

Thus, $\alpha_k(\overline{K}_{r,r}) = \omega_k(K_{r,r}) = 2k$.

Now $\omega_k(\overline{K}_{r,r}) \geq 2k - 1$ because each component H_i is complete and hence a k -plex of cardinality $2k - 1$. Suppose for contradiction that $\omega_k(\overline{K}_{r,r}) > 2k - 1$. Then there exists a k -plex $K \subseteq V$ such that $|K| = 2k$. If $|K \cap H_i| \leq k$, then

$$\deg_{\overline{K}_{r,r}[K]}(v) \leq k - 1 < k = |K| - k \text{ for all } v \in K \cap H_i.$$

This contradicts the definition of k -plex. Therefore, $|K \cap H_1| > k$ and $|K \cap H_2| > k$, which contradicts $|K| = 2k$. \square

Theorem 3. *Let $k > 1$. If $r = 2k - 1$, then $\overline{K}_{r,r}$ is not k -plex perfect.*

Proof. By Lemma 2, it suffices to show that $\chi_k(\overline{K}_{r,r}) \geq 2k$. Clearly, $\chi_k(\overline{K}_{r,r}) \geq \omega_k(\overline{K}_{r,r}) = 2k - 1$. Suppose for contradiction that $\chi_k(\overline{K}_{r,r}) = 2k - 1$. Lemma 1 implies the existence of a χ_k -optimal coloring S_1, \dots, S_m of $\overline{K}_{r,r}$ such that $|S_1| \geq k$. Therefore, $\omega_k(\overline{K}_{r,r}[S_1]) \geq k$. Furthermore, $\chi_k(\overline{K}_{r,r}) < 2k$ implies that all other sets S_i satisfy $|S_i| < k$. Notice that

$$2k - 1 = \chi_k(\overline{K}_{r,r}) = \sum_{i=1}^m \omega_k(\overline{K}_{r,r}[S_i]) \geq k + \sum_{i=2}^m \omega_k(\overline{K}_{r,r}[S_i]) = k + \sum_{i=2}^m |S_i|.$$

Consequently, $k - 1 \geq \sum_{i=2}^m |S_i|$. Now since the sets S_i partition V and $|V| = 4k - 2$,

$$|S_1| = |V| - \sum_{i=2}^m |S_i| \geq 3k - 1.$$

Therefore, $k > 1$ implies that $|S_1| \geq 3k - 1 > 2k$. This contradicts Lemma 2 because S_1 is a co- k -plex and $\alpha_k(\overline{K}_{r,r}) = 2k$. \square

Lovász's [16] replication lemma is a well-known result from the theory of perfect graphs. Replication of a vertex $v \in V$ corresponds to the following operation: create a new vertex v' and join it to v and all the neighbors of v . The replication lemma states that replication of a vertex in a perfect graph produces another perfect graph. However, for $k \geq 2$, replication of a vertex in a k -plex perfect graph does not necessarily produce another k -plex perfect graph.

Fix $k > 1$. Consider the edgeless graph G on two vertices v_1 and v_2 . G is a co- k -plex since $\Delta(G) = 0$. It follows that G is k -plex perfect. Construct G' by performing $2k - 2$ replication operations on each of v_1 and v_2 . This procedure creates $G' = \overline{K}_{r,r}$, which is not k -plex perfect by Theorem 3. Therefore, vertex replication does not preserve k -plex perfection. Theorem 3 also illustrates the following interesting property: G might not be k -plex perfect even if all components of G are k -plex perfect. This statement follows from Corollary 1 and Theorem 3.

The final topic of this section is a k -plex version of the Weak Perfect Graph Theorem [16]. The Weak Perfect Graph Theorem states that G is perfect if and only if \overline{G} is perfect. Theorems 2 and 3 together provide counterexamples for k -plex analogues of the Weak Perfect Graph Theorem for any $k \geq 2$.

3.2 A theorem of Erdős

In 1959, Erdős [10] showed that the difference $\chi_1(G) - \omega_1(G)$ can be arbitrarily large. More precisely, he showed that for every integer $r \geq 1$, there exists a graph G' with girth $g(G') > r$ and chromatic number $\chi(G') > r$. Observe that $g(G) > 3$ implies $\omega_1(G) \leq 2$. Therefore, the theorem establishes the existence of graphs with high chromatic number and low clique number.

Analogously, one might ask if the gap between $\chi_k(G)$ and $\omega_k(G)$ can also become arbitrarily large. This section uses the Erdős theorem to show that $\chi_k(G) - \omega_k(G)$ can be arbitrarily large. The proofs have been adapted from [8]. Let $\mathcal{G}_{n,p}$ be the random graph on n vertices where each edge exists with probability $0 \leq p \leq 1$. Let $q = 1 - p$.

Lemma 3. *Every co- k -plex S has at most $\frac{|S| \cdot (k-1)}{2}$ edges.*

Proof. $|E(S)| = \frac{1}{2} \cdot \sum_{v \in V(S)} \deg_{G[S]}(v) \leq \frac{1}{2} \cdot \sum_{v \in V(S)} (k-1) = \frac{|S| \cdot (k-1)}{2}$, where the inequality follows from the definition of co- k -plex. □

Lemma 4. For all integers $n \geq t \geq k + 1$, the probability that $G \in \mathcal{G}_{n,p}$ has a co- k -plex of size t is at most

$$P[\alpha_k(G) \geq t] \leq \binom{n}{t} q^{t(t-k)/2}.$$

Proof. Consider a fixed t -set $U \subseteq V$. By Lemma 3, the event that U is a co- k -plex is contained in the event that

$$|E(G[U])| \leq \frac{t(k-1)}{2}. \quad (3)$$

Thus, the probability of the latter is an upper bound on the probability of the former. Now (3) requires that at least $\binom{t}{2} - \frac{t(k-1)}{2}$ edges are missing. That is, (3) occurs with probability at most $q^{\binom{t}{2} - \frac{t(k-1)}{2}}$. The lemma follows from the fact that G contains $\binom{n}{t}$ t -sets U . \square

It is worth mentioning that if $t \leq \min\{k, n\}$, then every t -set is a co- k -plex, and Lemma 4 fails.

Theorem 4. Given any integer $r > k$, there exists a graph G with girth $g(G) > r$ and co- k -plex chromatic number $\chi_k(G) > r$.

Proof. For n large, suppose $t \geq \frac{n}{2r} > k$ and $(8r \ln n)n^{-1} \leq p \leq 1$. By Lemma 4,

$$P[\alpha_k \geq t] \leq \binom{n}{t} q^{t(t-k)/2} \leq n^t q^{t(t-k)/2} = (nq^{(t-k)/2})^t \leq (ne^{-p(t-k)/2})^t.$$

Therefore,

$$P[\alpha_k \geq t] \leq (ne^{-pt/2} e^{pk/2})^t \leq (ne^{-2(\ln n)} e^{k/2})^t = (n^{-1} e^{k/2})^t.$$

Now $\lim_{n \rightarrow \infty} n^{-1} e^{k/2} = 0$, so

$$P[\alpha_k \geq \frac{n}{2r}] < \frac{1}{2} \quad (4)$$

for sufficiently large n .

On the other hand, fix ϵ with $0 < \epsilon < 1/r$, and let $X(G)$ denote the number of cycles of length at most r in $G \in \mathcal{G}_{n,p}$. Erdős showed (see [8]) that for large n and $p = n^{\epsilon-1}$,

$$P[X \geq \frac{n}{2}] < \frac{1}{2}. \quad (5)$$

Finally, fix n large enough to satisfy (4), (5), and $n^{\epsilon-1} \geq (8r \ln n)n^{-1}$. Let $p = n^{\epsilon-1}$. There exists a $G \in \mathcal{G}_{n,p}$ such that $\alpha_k(G) < \frac{n}{2r}$ and G has less than $\frac{n}{2}$ cycles of length at most r . Construct the graph H by removing a vertex from each cycle of length at most r . Then $|H| \geq \frac{n}{2}$ and $g(H) > r$. Furthermore, $\alpha_k(H) \leq \alpha_k(G) < \frac{n}{2r}$ implies that any co- k -plex coloring of H requires more than r co- k -plex sets. Consequently, $\chi_k(H) > r$. \square

Corollary 2. *Given any integer $r > k + 2$, there exists a graph G with $\chi_k(G) > r$ and $\omega_k(G) < k + 2$.*

Proof. If $\omega_k(G) \geq k + 2$, then G contains a k -plex K of cardinality $k + 2$. Moreover, $\delta(G[K]) \geq 2$ by definition of k -plex. It follows that $G[K] \subseteq G$ contains a cycle of length at most $k + 2 = |K|$. Therefore, $g(G) > k + 2$ implies that $\omega_k(G) < k + 2$. The assertion now follows from Theorem 4. \square

4 Algorithms

This section develops algorithms for finding degree-bounded vertex partitions. Section 4.1 contains an exact $\bar{\chi}_k$ -coloring algorithm. Section 4.2 shows how to find χ_2 -optimal colorings using the $\bar{\chi}_2$ -coloring algorithm. Sections 4.1 and 4.2 both contain computational results. All implementations were run on a 2.2 GHz Dual-Core AMD Opteron processor with 3 GB of memory.

4.1 $\bar{\chi}_k$ -optimal coloring

In [15], Kubale and Jackowski present a generalized implicit enumeration algorithm for graph coloring which subsumes a number of previous combinatorial approaches [3, 4, 5, 14]. This section adapts the Kubale and Jackowski algorithm to find $\bar{\chi}_k$ -optimal colorings.

```

function implicitENUM( $G, n$ )
1.   $ub = n + 1; r = 1$ 
2.  loop
3.    FORWARDS( $r$ )
4.    BACKWARDS( $r$ )
5.    if  $r = 0$  then break
6.  repeat
end

function FORWARDS( $r$ )
7.  for  $i = r$  to  $n$ 
8.    reorder uncolored vertices  $v_i, \dots, v_n$ 
9.    if  $r = 1$  or  $r < i$  then determine  $FC(i)$ 
10.   if  $FC(i) = \emptyset$  then  $r = i$ ; return
11.    $C'(i) = \min(FC(i))$ 
12.  repeat
13.   $C = C'; ub = \max(C)$ 
14.   $r =$  least  $i$  such that  $C(i) = ub$ 
end

function BACKWARDS( $r$ )
15.   $CP = \{1, \dots, r - 1\}$ 
16.  while  $CP \neq \emptyset$ 
17.     $i = \max(CP); CP = CP - \{i\}$ 
18.     $FC(i) = FC(i) - \{C'(i)\}$ 
19.    if  $FC(i) \neq \emptyset$  then  $r = i$ ; return
20.  repeat
21.   $r = 0$ 
end

```

Figure 1: A generalized implicit enumeration algorithm [15].

Figure 4.1 contains the generalized implicit enumeration algorithm as given in [15]. Before running the algorithm, the vertex set of G is ordered (v_1, \dots, v_n) such that $\deg_G(v_i) \geq \deg_G(v_{i+1})$. The vertex ordering can either remain static or change dynamically throughout the algorithm. The array C' stores a partial co- k -plex coloring. The array C stores the incumbent co- k -plex coloring. For each $1 \leq i \leq n$, $FC(i)$ stores the set of feasible colors for v_i with respect to the current partial coloring C' . In other words, $FC(i)$ consists of partition classes S such that $S \cup \{v_i\}$ is a co- k -plex. CP is the set of current predecessors. These vertices are the candidates for backtracking.

The main difference between traditional graph coloring and $\bar{\chi}_k$ -coloring is the structure of the partition classes, so adapting the coloring algorithm in Figure 4.1 amounts to finding an appropriate definition for the set of feasible colors $FC(i)$. Given the partial co- k -plex coloring S_1, \dots, S_r , define

$$P(i) = \{j : S_j \cup \{v_i\} \text{ is not a co-}k\text{-plex}\} \cup \{ub\}.$$

The set of feasible colors is defined as $FC(i) = \{1, 2, \dots, \max_{j < i} C'(j) + 1\} - P(i)$. This definition forces each partition class to be a co- k -plex.

For $k = 2, 3, 4$, the algorithm was tested on a set of random graphs and a subset of the DIMACS coloring instances. The random graph $GN-P$ has N vertices and edge probability $\frac{P}{100}$. Table 1 contains a description of the test instances.

Tables 2 and 3 contain computational results obtained by running two versions of this algorithm. In the first version, $A1$, the vertex ordering remains static. For the second version, $A2$, the vertex ordering is dynamic. In determining the order, the algorithm always colors the vertex v_i such that $|P(i)|$ is maximum. Ties are broken by choosing the vertex of larger degree. This dynamic reordering is analogous to DSATUR [3].

Both algorithms appear to perform better on sparse graphs. $A2$ dominates $A1$ on all instances except for the graph *jean*. This suggests that it is worthwhile to use a dynamic

Table 1: Test instances

G	$ V $	$ E $	G	$ V $	$ E $
G20-10	20	21	G80-10	80	311
G20-30	20	56	G80-30	80	919
G20-50	20	98	G80-50	80	1583
G20-70	20	127	G80-70	80	2191
G20-90	20	170	G80-90	80	2876
G40-10	40	65	myciel3*	11	20
G40-30	40	223	myciel4*	23	71
G40-50	40	379	myciel5*	47	236
G40-70	40	539	myciel6*	95	755
G40-90	40	709	jean*	80	508
G60-10	60	173	queen5-5*	25	320
G60-30	60	546	queen6-6*	36	580
G60-50	60	902	anna*	138	986
G60-70	60	1244	homer*	561	3258
G60-90	60	1587	games120*	120	1276

* DIMACS graph

reordering scheme.

4.2 χ_2 -optimal coloring

Section 4.1 shows how traditional graph coloring algorithms can solve the $\bar{\chi}_k$ -coloring problem. However, these algorithms do not apply directly to χ_k -coloring since a χ_k -coloring algorithm must consider the weight $\omega_k(G[S_i])$ of each partition class in a co- k -plex coloring S_1, \dots, S_m .

This section focuses on the χ_2 -coloring problem. The proposed algorithm effectively reduces χ_2 -coloring to $\bar{\chi}_2$ -coloring. Lemma 1 implies that the set of compact co- k -plex colorings always contains a χ_k -optimal coloring. As a result, an algorithm can restrict the search for a χ_k -optimal solution by considering only compact co- k -plex colorings. For $k = 2$, Lemma 1 has an even stronger consequence. Recall that $\omega_2(G[C]) = \min\{2, |C|\} \in \{1, 2\}$ for any nonempty co-2-plex C [17].

Let Π_{min} be the set of compact $\bar{\chi}_2$ -optimal colorings. Define $\tilde{C}_1, \dots, \tilde{C}_m$ as follows: If possible, choose a co-2-plex coloring in Π_{min} with a deficient set, and let \tilde{C}_m be that deficient set. If no such co-2-plex coloring exists, choose an arbitrary co-2-plex coloring in Π_{min} .

Lemma 5. $\tilde{C}_1, \dots, \tilde{C}_m$ is a χ_2 -optimal coloring.

Table 2: $\bar{\chi}_k$ -coloring Algorithm A1

G	$\bar{\chi}_2(G)$	sec.	BBN	$\bar{\chi}_3(G)$	sec.	BBN	$\bar{\chi}_4(G)$	sec.	BBN
G20-10	2	0	5	2	0	4	2	0	6
G20-30	3	0	15	3	0	22	2	0	13
G20-50	4	0	32	3	0	552	3	0	99
G20-70	6	0	1300	4	0	647	4	0	7339
G20-90	7	0	17509	5	0	38043	5	29	2765462
G40-10	3	0	122	2	0	5	2	0	9
G40-30	4	0	3338	4	0	18399	3	0	401
G40-50	6	2	129385	5	12	1030246	4	5	390503
G40-70	9*	≥ 1000	78594881	7*	≥ 1000	72947725	6*	≥ 1000	63809697
G40-90	14*	≥ 1000	59897340	10*	≥ 1000	56745851	9*	≥ 1000	46934278
G60-10	3	0	12	3	0	526	2	0	2875
G60-30	6*	≥ 1000	82870726	5*	≥ 1000	67015032	4	234	18335449
G60-50	9*	≥ 1000	56966915	8*	≥ 1000	51671256	7*	≥ 1000	44186819
G60-70	14*	≥ 1000	37707967	12*	≥ 1000	38115506	11*	≥ 1000	35905870
G60-90	22*	≥ 1000	35322965	17*	≥ 1000	30052446	13*	≥ 1000	31400847
G80-10	3	80	4140070	3	0	3271	3	90	4184865
G80-30	7*	≥ 1000	43642550	7*	≥ 1000	34455942	5*	≥ 1000	46164683
G80-50	12*	≥ 1000	34269180	11*	≥ 1000	31980857	9*	≥ 1000	25000967
G80-70	17*	≥ 1000	30961357	15*	≥ 1000	27762778	12*	≥ 1000	23996017
G80-90	32*	≥ 1000	22305199	23*	≥ 1000	21832837	18*	≥ 1000	23367032
myciel3	2	0	4	2	0	6	2	0	8
myciel4	3	0	32	2	0	7	2	0	8
myciel5	4	0	24409	3	0	608	3	0	4888
myciel6	6*	≥ 1000	33247148	4*	≥ 1000	16332981	4*	≥ 1000	13670377
jean	5	4	488763	4	18	1803128	3	9	803339
anna	6	0	8525	4	8	320375	4	0	7613
queen5-5	5	1	56195	4	0	3393	3	0	4841
queen6-6	6	2	159863	4	43	3099994	4	3	170639
homer	9*	≥ 1000	933486	6*	≥ 1000	1166933	5*	≥ 1000	1199265
games120	5*	≥ 1000	30675394	4*	≥ 1000	11977879	4*	≥ 1000	10810550

* upper bound

Table 3: $\bar{\chi}_k$ -coloring Algorithm A2

G	$\bar{\chi}_2(G)$	sec.	BBN	$\bar{\chi}_3(G)$	sec.	BBN	$\bar{\chi}_4(G)$	sec.	BBN
G20-10	2	0	3	2	0	3	2	0	5
G20-30	3	0	4	3	0	15	2	0	4
G20-50	4	0	16	3	0	224	3	0	40
G20-70	6	0	952	4	0	499	4	0	5910
G20-90	7	0	15575	5	1	30893	5	66	2628085
G40-10	3	0	7	2	0	3	2	0	6
G40-30	4	0	45	4	0	687	3	0	51
G40-50	6	0	1688	5	4	36617	4	2	20637
G40-70	9	289	2850720	7*	≥ 1000	9495168	6*	≥ 1000	8173137
G40-90	14*	≥ 1000	10867861	10*	≥ 1000	11471697	9*	≥ 1000	12516863
G60-10	3	0	4	3	0	37	2	0	9
G60-30	6	58	212012	5*	≥ 1000	3885427	4	98	366802
G60-50	9*	≥ 1000	3660033	8*	≥ 1000	3587213	7*	≥ 1000	3902276
G60-70	13*	≥ 1000	3922763	11*	≥ 1000	3951519	11*	≥ 1000	4561966
G60-90	22*	≥ 1000	5849580	17*	≥ 1000	6241803	12*	≥ 1000	5917567
G80-10	3	0	1078	3	0	63	3	3	6717
G80-30	7*	≥ 1000	2233375	6*	≥ 1000	2168861	5*	≥ 1000	2354629
G80-50	11*	≥ 1000	1827244	10*	≥ 1000	2203187	8*	≥ 1000	2081951
G80-70	16*	≥ 1000	2210048	14*	≥ 1000	2418094	13*	≥ 1000	2841362
G80-90	31*	≥ 1000	3214420	23*	≥ 1000	4570884	18*	≥ 1000	4500137
myciel3	2	0	2	2	0	4	2	0	7
myciel4	3	0	9	2	0	11	2	0	14
myciel5	4	0	413	3	0	85	3	0	254
myciel6	5*	≥ 1000	2374317	4	921	1997558	4*	≥ 1000	1931171
jean	6*	≥ 1000	7052818	5*	≥ 1000	8001468	4*	≥ 1000	8976619
anna	6	0	1058	4	1	2183	4	1	5977
queen5-5	5	0	1838	4	0	461	3	0	416
queen6-6	6	1	8916	4	22	194542	4	3	25140
homer	9*	≥ 1000	314865	6*	≥ 1000	425440	5*	≥ 1000	503723
games120	5*	≥ 1000	3109430	4*	≥ 1000	2123862	3	0	606

* upper bound

Proof. Suppose for contradiction that $\tilde{C}_1, \dots, \tilde{C}_m$ is not χ_2 -optimal. $\tilde{C}_1, \dots, \tilde{C}_m$ is compact, so $|\tilde{C}_i| \geq 2$ for $i = 1, \dots, m - 1$. Therefore, $\sum_{i=1}^m \omega_2(G[\tilde{C}_i]) = 2(m - 1) + \min\{2, |\tilde{C}_m|\}$. Now consider a χ_2 -optimal coloring C_1, \dots, C_r . By Lemma 1, assume that C_1, \dots, C_r is compact. Therefore, $\chi_2(G) = \sum_{i=1}^r \omega_2(G[C_i]) = 2(r - 1) + \min\{2, |C_r|\}$, where C_r is the deficient set if one exists in C_1, \dots, C_r . Finally, observe that

$$2(m - 1) + \min\{2, |\tilde{C}_m|\} > \chi_2(G) = 2(r - 1) + \min\{2, |C_r|\},$$

which implies

$$\min\{2, |\tilde{C}_m|\} - \min\{2, |C_r|\} > 2(r - m).$$

Case 1: Suppose $r = m$. It follows that $\min\{2, |\tilde{C}_m|\} - \min\{2, |C_r|\} > 0$. Thus $\min\{2, |\tilde{C}_m|\} = 2$ and $\min\{2, |C_r|\} = 1$. In other words, \tilde{C}_m is not deficient; C_r is deficient; and C_1, \dots, C_r belongs to Π_{min} . This contradicts the choice of $\tilde{C}_1, \dots, \tilde{C}_m$.

Case 2: Suppose $r > m$. Then $\min\{2, |\tilde{C}_m|\} - \min\{2, |C_r|\} > 2$, a contradiction since $\min\{2, |\tilde{C}_m|\} \in \{1, 2\}$ and $\min\{2, |C_r|\} \in \{1, 2\}$.

□

Lemma 5 reduces the problem of finding a χ_2 -optimal coloring to that of finding an element in Π_{min} . This is desirable as the algorithm from Section 4.1 can solve the latter problem. The proposed algorithm consists of two steps, both of which make use of implicit enumeration.

The first step constructs a compact $\bar{\chi}_2$ -optimal coloring. The second step searches for a compact $\bar{\chi}_2$ -optimal coloring with one deficient set. If such a co-2-plex coloring exists, it is χ_2 -optimal by Lemma 5. If no such co-2-plex coloring exists, then the co-2-plex coloring from the first step is χ_2 -optimal by Lemma 5.

The first step uses the coloring algorithm exactly as described in Section 4.1. In the second step, for each $v \in V$, the algorithm uses implicit enumeration to search for a $\bar{\chi}_2(G) - 1$ coloring of $G - v$. If step two manages to find such a coloring in $G - v$, then v is a deficient

Table 4: χ_2 -coloring Algorithm

G	$\chi_2(G)$	sec.	BBN	G	$\chi_2(G)$	sec.	BBN
G20-10	4	0	23	G60-10	6	0	64
G20-30	6	0	24	G60-30	12	369	227480
G20-50	8	0	76	G80-10	6	4	1158
G20-70	12	1	3052	myciel3	4	0	13
G20-90	14	6	34475	myciel4	6	0	32
G40-10	5	0	7	myciel5	8	1	554
G40-30	8	0	165	anna	12	23	15548
G40-50	12	3	5958	queen5-5	10	1	2217
G40-70	18*	≥ 1000	1579353	queen6-6	12	9	14163

* upper bound

co-2-plex in a $\bar{\chi}_2$ -optimal coloring of G . A2 was used for the implicit enumeration. Table 4 contains computational results obtained by running the algorithm on the instances which were solved to $\bar{\chi}_2$ -optimality by A2.

Notice that most of these graphs satisfy $\chi_2(G) = 2 \cdot \bar{\chi}_2(G)$. These are exactly the graphs where step two of the algorithm failed to find a coloring with a deficient co-2-plex. However, the algorithm did find such a coloring for the graph G40-10.

5 Conclusion

This paper studies degree-bounded vertex partitions. Section 3 studies χ_k -optimal colorings, defines k -plex perfection, and offers examples of k -plex perfect graphs. It is also shown that many properties of graph perfection do not have k -plex analogues. Section 3.2 uses a theorem of Erdős to show that $\chi_k(G) - \omega_k(G)$ can become arbitrarily large.

This paper also derives algorithms for constructing degree-bounded vertex partitions. Section 4.1 offers a straightforward generalization of a traditional graph coloring algorithm. The resulting algorithm partitions the vertex set into the minimum number of co- k -plexes. Section 4.2 shows how to find χ_2 -optimal colorings by reducing the problem to that of finding $\bar{\chi}_2$ -optimal colorings. Lemma 5 represents a key step in the reduction. An interesting open problem is to determine if $\bar{\chi}_k$ -coloring can be used to solve χ_k -coloring problems for $k > 2$. Another avenue for future research is a polyhedral analysis of the co- k -plex coloring polytope.

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