COMPOSITION OF STABLE SET POLYHEDRA

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Abstract

Barahona and Mahjoub found a defining system of the stable set polytope for a graph with a cut-set of cardinality 2. We extend this result to cut-sets composed of a complete graph minus an edge and use the new theorem to derive a class of facets.

1 Preliminaries and Introduction

Let $G = (V, E)$ be a simple undirected graph. A set $S \subseteq V$ of pairwise nonadjacent vertices defines a stable set. Let $S := \{S \subseteq V \mid S \text{ is a stable set}\}$. Each $S \in S$ has an incidence vector $x^S \in \mathbb{R}^{|V|}$, where $x^S(v) = 1$ if $v \in S$ and $x^S(v) = 0$ otherwise. Let $P(G)$ be the convex hull of all $x^S$ such that $S \in S$. $P(G)$ is a full-dimensional polytope and has a unique (up to positive scalar multiples) minimal defining system. Let $V' \subset V$ and $E(V') := \{uv \in E \mid u, v \in V'\}$. The subgraph induced by $V'$ is $G[V'] := (V', E(V'))$. For $v \in V$, define $N_G(v) := \{u \in V \mid uv \in E\}$ and $\bar{N}_G(v) := N_G(v) \cup \{v\}$. A complete graph consists of pairwise adjacent vertices. A vertex set $K \subseteq V$ is complete whenever $G[K]$ is a complete subgraph. A maximal complete subgraph defines a clique. A vertex $v \in V$ is simplicial if $G[N(v)]$ is complete.

A cut-set $C \subset V$ decomposes $G$ into a pair of proper subgraphs $(G_1, G_2)$ such that $C = V(G_1) \cap V(G_2)$ and all paths from $G_1$ to $G_2$ intersect $C$. Chvátal [2] showed that the union of defining systems of $P(G_1)$ and $P(G_2)$ defines $P(G)$ when $G[C]$ is a clique. Barahona and Mahjoub [1] defined $P(G)$ based on systems related to $P(G_1)$ and $P(G_2)$ when $|C| \leq 2$. We extend this result to the case where $G[C]$ is a complete graph minus an edge.

Section 2 contains results necessary to extend Barahona and Mahjoub’s theorem. Section 3 generalizes their theorem. Section 3 refers to results from [1]. Section 4 applies the new theorem to derive a class of facets for the stable set polytope called diamonds. Section 5 summarizes the results and discusses future research.

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2 Support Graphs

Suppose $G$ has a cut-set $C$ consisting of a nonadjacent pair of vertices. To obtain a defining system for $P(G)$, Barahona and Mahjoub [1] attach to $C$ a new set of vertices $\{w_i\}$. This augmentation defines a graph $\tilde{G}$. $P(\tilde{G})$ has a facet which projects along the subspace of $\{w_i\}$ variables to define $P(G)$. We generalize this method to the case where $G[C]$ is a complete graph minus an edge. Section 3 analyzes the decomposition of $\tilde{G}$ into the pair $(\tilde{G}_1, \tilde{G}_2)$. Here, we determine how the support graphs of facets for $P(\tilde{G}_k)$ interact with the $\{w_i\}$ vertices.

Let $a^T x \leq b$ be a nontrivial facet of $P(G)$. Nontriviality implies $b > 0$ and $a_v \geq 0$ for all $v \in V$. In this section, all facets are assumed to be nontrivial. Define the following sets:

$$V_a := \{v \in V \mid a_v > 0\} \text{ and } \mathcal{F}_a := \{S \in \mathcal{S} \mid a^T x^S = b\}.$$  

The support graph of $a^T x \leq b$ is defined as $G_a := G[V_a]$, the subgraph induced by $V_a$.

**Remark 1.** Given a facet $a^T x \leq b$, $\mathcal{F}_a$ consists of maximal stable sets in $G_a$.

In Section 3, we partition inequalities based on their intersection with the set $\{w_i\}$. Lemma 1 reduces the number of partition sets. Recall that since $P(G)$ is full-dimensional, the sets $S \in \mathcal{F}_a$ collectively satisfy no equations other than scalar multiples of $a^T x = b$.

**Lemma 1.** If $a^T x \leq b$ is a non-clique facet, then $G_a$ contains no simplicial vertex.

**Proof.** Suppose $v \in V_a$ is simplicial in $G_a$. Then $K := N_{G_a}(v)$ is a clique and there exists an $S \in \mathcal{F}_a$ such that $S \cap K = \emptyset$. Otherwise, $\sum_{v \in K} x^S(v) = 1$ for all $S' \in \mathcal{F}_a$, a contradiction because $a^T x \leq b$ is not a clique inequality. Observe that $S$ is not a maximal stable set in $G_a$, since $S \cup \{v\}$ is a feasible stable set. This contradicts Remark 1. $\square$

Suppose $G = (G_1, G_2)$ has a cut-set $C$ where $G[C]$ is a complete graph minus an edge. Notice $G[C]$ has a stable set $\{u, v\}$. For $k \in \{1, 2\}$, add the $\{w_i\}$ vertices to $G_k$ such that $N_{G_k}(w_1) = \{w_2\} \cup (C \setminus \{u\})$, $N_{G_k}(w_2) = \{w_1, u\}$, and $N_{G_k}(w_3) = C$. See Figure 1 for the augmented graph $\tilde{G}_k$. The heavy edges denote joins (see [4]). For example, the edge between $u$ and $C \setminus \{u, v\}$ indicates that $u$ is adjacent to every vertex in $C \setminus \{u, v\}$.

**Lemma 2.** Let $u, v \in C$ be nonadjacent and $\tilde{C} := C \cup \{w_1, w_2, w_3\}$. For $k \in \{1, 2\}$,

$$F_k := \{x \in P(\tilde{G}_k) \mid \sum_{z \in \tilde{C}} x(z) = 2\}$$

is a facet for $P(\tilde{G}_k)$. Moreover, no other facet contains all the vertices of $\tilde{C}$ in its support.
Proof. We show that $F_k$ is a facet for $P(\tilde{G}_k[\tilde{C}])$ by building a full-rank $|\tilde{C}| \times |\tilde{C}|$ matrix whose columns are incidence vectors of all stable sets which lie on $F_k$. See Figure 2. The first three rows correspond to $w_1$, $w_2$, and $w_3$. The last rows correspond to $u$ and $C \setminus \{u\}$, respectively. Let $\tilde{I}_v$ be the $(|C|-1)$-dimensional column vector with a 1 in row $v$ and 0’s elsewhere.

We now lift the inequality $\sum_{z \in \tilde{C}} x(z) \leq 2$ to a facet of $P(\tilde{G}_k)$. Since all maximal stable sets $J$ in $\tilde{G}_k$ satisfy $|J \cap \tilde{C}| = 2$, the lifting coefficients for vertices in $V(\tilde{G}_k) \setminus \tilde{C}$ are zero. Thus, the inequality is a facet of $P(\tilde{G}_k)$. Suppose another facet $a^T x \leq b$ contains all vertices of $\tilde{C}$ in its support. By Remark 1, $\sum_{z \in \tilde{C}} x^{S'}(z) = 2$ for all $S' \in F_a$. It follows that $F_k$ coincides with the face induced by $a^T x \leq b$. \hfill \Box

\[
\begin{bmatrix}
1 & \bar{0}^T & 1 & 0 & 0 \\
0 & \bar{I}^T & 0 & 1 & 0 \\
0 & \bar{0}^T & 1 & 1 & 0 \\
1 & \bar{0}^T & 0 & 0 & 1 \\
\bar{0} & I^{(|C|-1)\times(|C|-1)} & \bar{0} & \bar{0} & \tilde{I}_v
\end{bmatrix}
\]

Figure 2: The matrix $A$.

Given a defining system for a polytope, the process of projecting along a subspace of variables, say $w_1$ and $w_2$, is less complicated if the coefficients of $w_1$ and $w_2$ are binary. The following lemma allows the defining systems encountered in Section 3 to be put in this form.

**Lemma 3** (Mahjoub [3]). Given a facet $a^T x \leq b$, let $w_1, w_2 \in V_a$ be adjacent vertices in $G_a$. If $w_1$ is simplicial in $G_a - w_2$ and $w_2$ is simplicial in $G_a - w_1$, then $a_{w_1} = a_{w_2}$.

Lemma 3 implies that $a_{w_1} = a_{w_2}$ in any nontrivial facet containing both $w_1$ and $w_2$ in its support. As a result, scaling these inequalities by $(1/a_{w_1}) = (1/a_{w_2})$ will produce inequalities where both variables have binary coefficients.

### 3 Composition of Stable Set Polyhedra

This section offers a straightforward extension of techniques developed by Barahona and Mahjoub. We will refer to results from [1]. Let $G = (G_1, G_2)$ have a cut-set $C$ where $G[C]$ is a complete graph minus an edge. Construct the augmented graph $\tilde{G}$ by adding a new set of vertices $\{w_i\}$ to $C$, as in Section 2. Define $\tilde{C} := C \cup \{w_i\}$. $P(\tilde{G})$ has a facet $F = \{x \in P(\tilde{G}) \mid \sum_{z \in \tilde{C}} x(z) = 2\}$ such that

$$P(G) = \text{proj}_{w_1, w_2, w_3} \{F\} = \{x \in \mathbb{R}^{|G|} \mid \exists \ w \in \mathbb{R}^3 \text{ s.t. } (x, w) \in F\}.$$ 

The set $\tilde{C}$ decomposes $\tilde{G}$ into the pair $(\tilde{G}_1, \tilde{G}_2)$. In Section 2, it was shown that $P(\tilde{G}_k)$ has a facet $F_k$ for $k \in \{1, 2\}$. 

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Lemma 4 (Barahona and Mahjoub [1]). The facet $F$ is defined by the union of the systems that define $F_1$ and $F_2$.

Lemma 4 relies on the existence of a full-rank, square matrix of all incidence vectors for stable sets on $F$, $F_1$, and $F_2$. The matrix $A$ constructed in the proof of Lemma 2 (see Figure 2) implies that this lemma holds for the class of cut-sets $C$ we are analyzing. In order to find a defining system for $F$, consider the defining system for $P(G_k)$ (other than clique inequalities involving the $\{w_i\}$ variables). Recall from Section 2 that the support of $a^T x \leq b$ is denoted by $V_a$. Lemma 1 and Lemma 2 imply that the facet-defining inequalities can be partitioned into three sets $I^k_1, I^k_2, I^k_3$ defined as follows:

$$I^k_1 := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \emptyset\}$$
$$I^k_2 := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \{w_1, w_2\}\}$$
$$I^k_3 := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \{w_3\}\}.$$

Let $V_k = V(G_k)$. Lemma 3 and Lemma 4 imply that the defining system of $F$ can be written as follows, $k \in \{1, 2\}$:

1. $\sum_{j \in V_k} a_{ij}^k x(j) \leq b_i^k$, for all $i \in I^k_1$
2. $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_1) + x(w_2) \leq b_i^k$, for all $i \in I^k_2$
3. $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_3) \leq b_i^k$, for all $i \in I^k_3$
4. $\sum_{j \in C \setminus u} x(j) + x(w_1) \leq 1$
5. $\sum_{j \in C \setminus u} x(j) + x(w_3) \leq 1$
6. $\sum_{j \in C \setminus u} x(j) + x(w_3) \leq 1$
7. $x(u) + x(w_2) \leq 1$
8. $x(w_1) + x(w_2) \leq 1$
9. $\sum_{j \in \tilde{C}} x(j) = 2$
10. $x(j) \geq 0$, for all $j \in \tilde{V}_k$.

The projection of this system along the subspace of the $\{w_i\}$ variables is the polytope $P(G)$. To define $P(G)$, we proceed exactly as in [1].

Theorem 1. The polytope $P(G)$ is defined by the union of defining systems for $P(G_1)$ and $P(G_2)$, the non-negativity constraints, and the following facet-defining mixed inequalities:

$$\sum_{j \in V_k} a_{ij}^k x(j) + \sum_{j \in V_l} a_{ij}^l x(j) - \sum_{j \in \tilde{C}} x(j) \leq b_i^k + b_i^l - 2 \quad \text{for} \quad k = 1, 2; l = 1, 2; k \neq l; i \in I^k_2; r \in I^l_3.$$

Proof. See Theorem 3.5 and Corollary 3.7 in [1].
4 Diamonds

This section uses Theorem 1 to derive a class of facets for $P(G)$. Let $K_1, ..., K_6$ be sets of vertices such that each $K_i$ is nonempty and complete. The graph $G$ shown in Figure 3 is a member of a class of graphs which we call diamonds. The heavy edges denote joins. For example, an edge between $K_i$ and $K_j$ indicates that $G[K_i \cup K_j]$ is complete. The size of the diamond is equal to the number of sets $K_i$. The diamond in Figure 3 has size 6, and $\sum_{z \in V} x(z) \leq 3$ induces a facet for $P(G)$. In general, facet-inducing diamonds have size $2n$ (where $n > 1$), a vertex $u$ such that $N_G(u) = \bigcup_{i=1}^{2n} K_i$, and a path $P = p_1p_2...p_{2n-2}$ attached to the sets $K_1, ..., K_{2n}$ as shown in Figure 3.

![Figure 3: A diamond of size 6.](image)

Theorem 2. Let $n > 1$. If a diamond $G$ has size $2n$, then $\sum_{z \in V} x(z) \leq n$ induces a facet for $P(G)$.

Proof. The proof is by induction on $n$.

**Base case ($n = 2$):** Choose $v \in K_1$ and $w \in K_4$. The diamond of size 4 has a 5-hole on the vertex set $\{p_1, p_2, w, u, v\}$. Moreover, the odd-hole inequality can be lifted to include all vertices in $\bigcup_{i=1}^{4} K_i$. This implies that $\sum_{z \in V} x(z) \leq 2$ induces a facet for $P(G)$ as claimed.

**Induction step ($n > 2$):** Suppose the theorem holds for all diamonds of even size less than $2n$. The diamond of size $2n$ has a cut-set $C = K_{2n-3} \cup \{u, p_{2n-4}\}$ which can be constructed by removing an edge from a complete graph. Therefore, we apply Theorem 1. Figure 4 shows subgraphs of the pair $(\tilde{G}_1, \tilde{G}_2)$. Let $V_1' = V(\tilde{G}_1) \setminus \{w_1, w_2\}$ and $V_2' = V(\tilde{G}_2) \setminus \{w_3\}$. The graph on the left is a diamond of size $2n - 2$. By induction,

$$\sum_{z \in V_1'} x(z) \leq n - 1 \quad (1)$$

is a facet for $P(\tilde{G}_1)$. $\tilde{G}_2$ has an odd-hole inequality which lifts to obtain that

$$\sum_{z \in V_2'} x(z) \leq 3 \quad (2)$$

is a facet for $P(\tilde{G}_2)$.

Notice that inequality (1) $\in I_3^1$ and inequality (2) $\in I_2^2$. Theorem 1 gives the following facet-defining mixed inequality for $P(G)$:

$$\sum_{z \in V(\tilde{G}_1)} x(z) + \sum_{z \in V(\tilde{G}_2)} x(z) - \sum_{z \in C} x(z) \leq n - 1 + 3 - 2$$
Upon simplifying, we obtain that $\sum_{z \in V} x(z) \leq n$ is a facet for $P(G)$ as claimed.

Theorem 2 fails when the diamond has size that is odd and at least three. To see this, let $G$ be the diamond of size 3 shown in Figure 5. $G$ is perfect and not a clique, so $G$ is not a support graph for any facet of $P(G)$. Now let $G$ be the diamond of size 5 also shown in Figure 5. If $G$ is the support graph of a facet, then there must exist $4 + \sum_{i=1}^{5} |K_i|$ affinely independent maximal stable sets satisfying some equation. However, no such set exists. It follows by induction that a diamond of odd size is not a support graph for any facet of $P(G)$.

5 Conclusions and Future Work

This paper generalized a theorem of Barahona and Mahjoub concerning the composition of stable set polyhedra. The new result was applied to derive a class of facets called diamonds. Future research includes using composition to derive other classes of facets. An extension of Theorem 1 to more general cut-sets would also be beneficial since composition can be applied recursively. In other words, $G$ can be decomposed into subgraphs $G_1, ..., G_m$ such that the defining system for each $P(G_i)$ is known. For example, decompose $G$ into a set of perfect graphs. The defining systems for each $P(G_i)$ can then be composed to define $P(G)$. Another idea is to construct $P(G)$ starting from the leaves of a tree or branch decomposition. These approaches have the potential to characterize the stable set polytope for graphs which admit a structured decomposition, but they require a more general form of Theorem 1.

Generalizing Theorem 1 might require techniques different from the lift and project method of Barahona and Mahjoub. Finding a $\tilde{G}_k$ and $\tilde{F}_k$ with the correct structure ap-
pears to be difficult. A subtle requirement is that $\tilde{G}_k[\tilde{C}]$ has to have exactly $|\tilde{C}|$ affinely independent maximum stable sets. Otherwise, the matrix $A$ is not invertible and Lemma 4 fails. Without this restriction, Theorem 1 would have held for any cut-set which partitions into two cliques.

There exist many graphs $\tilde{G}_k[\tilde{C}]$ with exactly $|\tilde{C}|$ affinely independent maximum stable sets, but the inequalities which define $F_k$ must also involve the $w_i$ vertices in a structured way. This structure would most likely involve extending the results of Section 2 to prevent the projection step from becoming too complicated.

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References


