COMPOSITION OF STABLE SET POLYHEDRA

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Abstract

Barahona and Mahjoub found a defining system of the stable set polytope for a graph with a cut-set of cardinality 2. We extend this result to cut-sets composed of a complete graph minus an edge and use the new theorem to derive a class of facets.

1 Preliminaries and Introduction

Let $G = (V, E)$ be a simple undirected graph. A set $S \subseteq V$ of pairwise nonadjacent vertices defines a stable set. Let $\mathcal{S} := \{S \subseteq V \mid S \text{ is a stable set}\}$. Each $S \in \mathcal{S}$ has an incidence vector $x^S \in \mathbf{R}^{|V|}$, where $x^S(v) = 1$ if $v \in S$ and $x^S(v) = 0$ otherwise. Let $P(G)$ be the convex hull of all x^S such that $S \in \mathcal{S}$. $P(G)$ is a full-dimensional polytope and has a unique (up to positive scalar multiples) minimal defining system. Let $V' \subset V$ and $E(V') := \{uv \in E \mid u, v \in V'\}$. The subgraph induced by V' is $G[V'] := (V', E(V'))$. For $v \in V$, define $N_G(v) := \{u \in V \mid uv \in E\}$ and $\bar{N}_G(v) := N_G(v) \cup \{v\}$. A complete graph consists of pairwise adjacent vertices. A vertex set $K \subseteq V$ is complete whenever $G[K]$ is a complete subgraph. A maximal complete subgraph defines a clique. A vertex $v \in V$ is *simplicial* if $G[N(v)]$ is complete.

A cut-set $C \subset V$ decomposes G into a pair of proper subgraphs (G_1, G_2) such that $C = V(G_1) \cap V(G_2)$ and all paths from G_1 to G_2 intersect C. Chvátal [2] showed that the union of defining systems of $P(G_1)$ and $P(G_2)$ defines $P(G)$ when $G[C]$ is a clique. Barahona and Mahjoub [1] defined $P(G)$ based on systems related to $P(G_1)$ and $P(G_2)$ when $|C| \leq 2$. We extend this result to the case where $G[C]$ is a complete graph minus an edge.

Section 2 contains results necessary to extend Barahona and Mahjoub's theorem. Section 3 generalizes their theorem. Section 3 refers to results from [1]. Section 4 applies the new theorem to derive a class of facets for the stable set polytope called diamonds. Section 5 summarizes the results and discusses future research.

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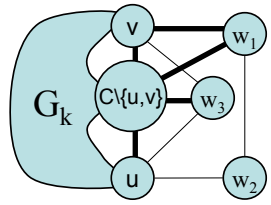

Figure 1: The augmented graph $\tilde{G}_k$.

2 Support Graphs

Suppose G has a cut-set C consisting of a nonadjacent pair of vertices. To obtain a defining system for $P(G)$, Barahona and Mahjoub [1] attach to C a new set of vertices $\{w_i\}$. This augmentation defines a graph $\tilde{G}$. $P(\tilde{G})$ has a facet which projects along the subspace of $\{w_i\}$ variables to define $P(G)$. We generalize this method to the case where $G[C]$ is a complete graph minus an edge. Section 3 analyzes the decomposition of $\tilde{G}$ into the pair $(\tilde{G}_1, \tilde{G}_2)$. Here, we determine how the support graphs of facets for $P(\tilde{G}_k)$ interact with the $\{w_i\}$ vertices.

Let $a^T x \leq b$ be a nontrivial facet of $P(G)$. Nontriviality implies $b > 0$ and $a_v \geq 0$ for all $v \in V$. In this section, all facets are assumed to be nontrivial. Define the following sets:

$$V_a := \{v \in V \mid a_v > 0\} \text{ and } \mathcal{F}_a := \{S \in \mathcal{S} \mid a^T x^S = b\}.$$

The *support graph* of $a^T x \leq b$ is defined as $G_a := G[V_a]$, the subgraph induced by V_a.

Remark 1. *Given a facet $a^T x \leq b$, $\mathcal{F}_a$ consists of maximal stable sets in G_a.*

In Section 3, we partition inequalities based on their intersection with the set $\{w_i\}$. Lemma 1 reduces the number of partition sets. Recall that since $P(G)$ is full-dimensional, the sets $S \in \mathcal{F}_a$ collectively satisfy no equations other than scalar multiples of $a^T x = b$.

Lemma 1. *If $a^T x \leq b$ is a non-clique facet, then G_a contains no simplicial vertex.*

Proof. Suppose $v \in V_a$ is simplicial in G_a. Then $K := \bar{N}_{G_a}(v)$ is a clique and there exists an $S \in \mathcal{F}_a$ such that $S \cap K = \emptyset$. Otherwise, $\sum_{v \in K} x^{S'}(v) = 1$ for all $S' \in \mathcal{F}_a$, a contradiction because $a^T x \leq b$ is not a clique inequality. Observe that S is not a maximal stable set in G_a, since $S \cup \{v\}$ is a feasible stable set. This contradicts Remark 1. □

Suppose $G = (G_1, G_2)$ has a cut-set C where $G[C]$ is a complete graph minus an edge. Notice $G[C]$ has a stable set $\{u, v\}$. For $k \in \{1, 2\}$, add the $\{w_i\}$ vertices to G_k such that $N_{\tilde{G}_k}(w_1) = \{w_2\} \cup (C \setminus \{u\})$, $N_{\tilde{G}_k}(w_2) = \{w_1, u\}$, and $N_{\tilde{G}_k}(w_3) = C$. See Figure 1 for the augmented graph $\tilde{G}_k$. The heavy edges denote joins (see [4]). For example, the edge between u and $C \setminus \{u, v\}$ indicates that u is adjacent to every vertex in $C \setminus \{u, v\}$.

Lemma 2. *Let $u, v \in C$ be nonadjacent and $\tilde{C} := C \cup \{w_1, w_2, w_3\}$. For $k \in \{1, 2\}$,*

$$F_k := \{x \in P(\tilde{G}_k) \mid \sum_{z \in \tilde{C}} x(z) = 2\}$$

is a facet for $P(\tilde{G}_k)$. Moreover, no other facet contains all the vertices of $\tilde{C}$ in its support.

Proof. We show that F_k is a facet for $P(\tilde{G}_k[\tilde{C}])$ by building a full-rank $|\tilde{C}| \times |\tilde{C}|$ matrix whose columns are incidence vectors of all stable sets which lie on F_k. See Figure 2. The first three rows correspond to w_1, w_2, and w_3. The last rows correspond to u and $C \setminus \{u\}$, respectively. Let $\vec{1}_v$ be the ($|C|$-1)-dimensional column vector with a 1 in row v and 0's elsewhere.

We now lift the inequality $\sum_{z \in \tilde{C}} x(z) \leq 2$ to a facet of $P(\tilde{G}_k)$. Since all maximal stable sets J in $\tilde{G}_k$ satisfy $|J \cap \tilde{C}| = 2$, the lifting coefficients for vertices in $V(\tilde{G}_k) \setminus \tilde{C}$ are zero. Thus, the inequality is a facet of $P(\tilde{G}_k)$. Suppose another facet $a^T x \leq b$ contains all vertices of $\tilde{C}$ in its support. By Remark 1, $\sum_{z \in \tilde{C}} x^{S'}(z) = 2$ for all $S' \in \mathcal{F}_a$. It follows that F_k coincides with the face induced by $a^T x \leq b$. $\square$

$$\begin{bmatrix} 1 & \vec{0}^T & 1 & 0 & 0 \\ 0 & \vec{1}^T & 0 & 1 & 0 \\ 0 & \vec{0}^T & 1 & 1 & 0 \\ 1 & \vec{0}^T & 0 & 0 & 1 \\ \vec{0} & I^{(|C|-1)\times(|C|-1)} & \vec{0} & \vec{0} & \vec{1}_v \end{bmatrix}$$

Figure 2: The matrix A.

Given a defining system for a polytope, the process of projecting along a subspace of variables, say w_1 and w_2, is less complicated if the coefficients of w_1 and w_2 are binary. The following lemma allows the defining systems encountered in Section 3 to be put in this form.

Lemma 3 (Mahjoub [3])**.** *Given a facet $a^T x \leq b$, let $w_1, w_2 \in V_a$ be adjacent vertices in G_a. If w_1 is simplicial in $G_a - w_2$ and w_2 is simplicial in $G_a - w_1$, then $a_{w_1} = a_{w_2}$.*

Lemma 3 implies that $a_{w_1} = a_{w_2}$ in any nontrivial facet containing both w_1 and w_2 in its support. As a result, scaling these inequalities by $(1/a_{w_1}) = (1/a_{w_2})$ will produce inequalities where both variables have binary coefficients.

3 Composition of Stable Set Polyhedra

This section offers a straightforward extension of techniques developed by Barahona and Mahjoub. We will refer to results from [1]. Let $G = (G_1, G_2)$ have a cut-set C where $G[C]$ is a complete graph minus an edge. Construct the augmented graph $\tilde{G}$ by adding a new set of vertices $\{w_i\}$ to C, as in Section 2. Define $\tilde{C} := C \cup \{w_i\}$. $P(\tilde{G})$ has a facet $F = \{x \in P(\tilde{G}) \mid \sum_{z \in \tilde{C}} x(z) = 2\}$ such that

$$P(G) = proj_{w_1,w_2,w_3}\{F\} = \{x \in \mathbf{R}^{|G|} \mid \exists\ w \in \mathbf{R}^3 \text{ s.t. } (x, w) \in F\}.$$

The set $\tilde{C}$ decomposes $\tilde{G}$ into the pair $(\tilde{G}_1, \tilde{G}_2)$. In Section 2, it was shown that $P(\tilde{G}_k)$ has a facet F_k for $k \in \{1, 2\}$.

Lemma 4 (Barahona and Mahjoub [1]). *The facet F is defined by the union of the systems that define F_1 and F_2.*

Lemma 4 relies on the existence of a full-rank, square matrix of all incidence vectors for stable sets on F, F_1, and F_2. The matrix A constructed in the proof of Lemma 2 (see Figure 2) implies that this lemma holds for the class of cut-sets C we are analyzing. In order to find a defining system for F, consider the defining system for $P(\tilde{G}_k)$ (other than clique inequalities involving the $\{w_i\}$ variables). Recall from Section 2 that the support of $a^T x \leq b$ is denoted by V_a. Lemma 1 and Lemma 2 imply that the facet-defining inequalities can be partitioned into three sets I_1^k, I_2^k, I_3^k defined as follows:

$$I_1^k := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \emptyset\}$$
$$I_2^k := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \{w_1, w_2\}\}$$
$$I_3^k := \{a_i^T x \leq b_i \mid V_{a_i} \cap \{w_1, w_2, w_3\} = \{w_3\}\}.$$

Let $V_k = V(G_k)$. Lemma 3 and Lemma 4 imply that the defining system of F can be written as follows, $k \in \{1, 2\}$:

1. $\sum_{j \in V_k} a_{ij}^k x(j) \leq b_i^k$, for all $i \in I_1^k$
2. $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_1) + x(w_2) \leq b_i^k$, for all $i \in I_2^k$
3. $\sum_{j \in V_k} a_{ij}^k x(j) + x(w_3) \leq b_i^k$, for all $i \in I_3^k$
4. $\sum_{j \in C \setminus u} x(j) + x(w_1) \leq 1$
5. $\sum_{j \in C \setminus u} x(j) + x(w_3) \leq 1$
6. $\sum_{j \in C \setminus v} x(j) + x(w_3) \leq 1$
7. $x(u) + x(w_2) \leq 1$
8. $x(w_1) + x(w_2) \leq 1$
9. $\sum_{j \in \tilde{C}} x(j) = 2$
10. $x(j) \geq 0$, for all $j \in \tilde{V}_k$.

The projection of this system along the subspace of the $\{w_i\}$ variables is the polytope $P(G)$. To define $P(G)$, we proceed exactly as in [1].

Theorem 1. *The polytope $P(G)$ is defined by the union of defining systems for $P(G_1)$ and $P(G_2)$, the non-negativity constraints, and the following facet-defining mixed inequalities:*

$$\sum_{j \in V_k} a_{ij}^k x(j) + \sum_{j \in V_l} a_{rj}^l x(j) - \sum_{j \in C} x(j) \leq b_i^k + b_r^l - 2 \quad for\ k = 1, 2; l = 1, 2; k \neq l; i \in I_2^k; r \in I_3^l.$$

Proof. See Theorem 3.5 and Corollary 3.7 in [1]. □

4 Diamonds

This section uses Theorem 1 to derive a class of facets for $P(G)$. Let $K_1, ..., K_6$ be sets of vertices such that each K_i is nonempty and complete. The graph G shown in Figure 3 is a member of a class of graphs which we call diamonds. The heavy edges denote joins. For example, an edge between K_i and K_j indicates that $G[K_i \cup K_j]$ is complete. The size of the diamond is equal to the number of sets K_i. The diamond in Figure 3 has size 6, and $\sum_{z \in V} x(z) \leq 3$ induces a facet for $P(G)$. In general, facet-inducing diamonds have size $2n$ (where $n > 1$), a vertex u such that $N_G(u) = \bigcup_{i=1}^{2n} K_i$, and a path $P = p_1 p_2 ... p_{2n-2}$ attached to the sets $K_1, ..., K_{2n}$ as shown in Figure 3.

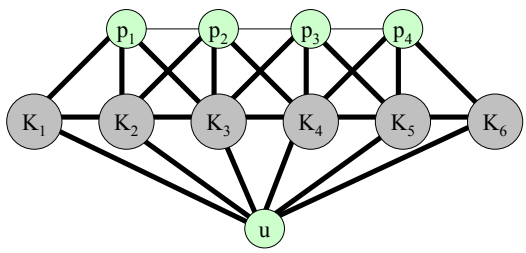

Figure 3: A diamond of size 6.

Theorem 2. *Let $n > 1$. If a diamond G has size $2n$, then $\sum_{z \in V} x(z) \leq n$ induces a facet for $P(G)$.*

Proof. The proof is by induction on n.

Base case ($n = 2$): Choose $v \in K_1$ and $w \in K_4$. The diamond of size 4 has a 5-hole on the vertex set $\{p_1, p_2, w, u, v\}$. Moreover, the odd-hole inequality can be lifted to include all vertices in $\bigcup_{i=1}^{4} K_i$. This implies that $\sum_{z \in V} x(z) \leq 2$ induces a facet for $P(G)$ as claimed.

Induction step ($n > 2$): Suppose the theorem holds for all diamonds of even size less than $2n$. The diamond of size $2n$ has a cut-set $C = K_{2n-3} \cup \{u, p_{2n-4}\}$ which can be constructed by removing an edge from a complete graph. Therefore, we apply Theorem 1. Figure 4 shows subgraphs of the pair $(\tilde{G}_1, \tilde{G}_2)$. Let $V_1' = V(\tilde{G}_1) \setminus \{w_1, w_2\}$ and $V_2' = V(\tilde{G}_2) \setminus \{w_3\}$. The graph on the left is a diamond of size $2n - 2$. By induction,

$$\sum_{z \in V_1'} x(z) \leq n - 1 \tag{1}$$

is a facet for $P(\tilde{G}_1)$. $\tilde{G}_2$ has an odd-hole inequality which lifts to obtain that

$$\sum_{z \in V_2'} x(z) \leq 3 \tag{2}$$

is a facet for $P(\tilde{G}_2)$.

Notice that inequality (1) $\in I_3^1$ and inequality (2) $\in I_2^2$. Theorem 1 gives the following facet-defining mixed inequality for $P(G)$:

$$\sum_{z \in V(G_1)} x(z) + \sum_{z \in V(G_2)} x(z) - \sum_{z \in C} x(z) \leq n - 1 + 3 - 2$$

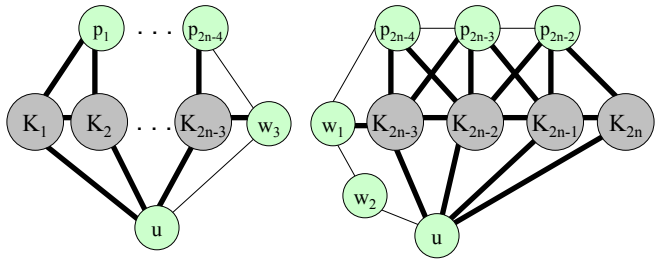

Figure 4: Subgraphs of $\tilde{G}_1$ and $\tilde{G}_2$.

Upon simplifying, we obtain that $\sum_{z \in V} x(z) \leq n$ is a facet for $P(G)$ as claimed.

□

Theorem 2 fails when the diamond has size that is odd and at least three. To see this, let G be the diamond of size 3 shown in Figure 5. G is perfect and not a clique, so G is not a support graph for any facet of $P(G)$. Now let G be the diamond of size 5 also shown in Figure 5. If G is the support graph of a facet, then there must exist $4 + \sum_{i=1}^{5} |K_i|$ affinely independent maximal stable sets satisfying some equation. However, no such set exists. It follows by induction that a diamond of odd size is not a support graph for any facet of $P(G)$.

5 Conclusions and Future Work

This paper generalized a theorem of Barahona and Mahjoub concerning the composition of stable set polyhedra. The new result was applied to derive a class of facets called diamonds. Future research includes using composition to derive other classes of facets. An extension of Theorem 1 to more general cut-sets would also be beneficial since composition can be applied recursively. In other words, G can be decomposed into subgraphs $G_1, ..., G_m$ such that the defining system for each $P(G_i)$ is known. For example, decompose G into a set of perfect graphs. The defining systems for each $P(G_i)$ can then be composed to define $P(G)$. Another idea is to construct $P(G)$ starting from the leaves of a tree or branch decomposition. These approaches have the potential to characterize the stable set polytope for graphs which admit a structured decomposition, but they require a more general form of Theorem 1.

Generalizing Theorem 1 might require techniques different from the lift and project method of Barahona and Mahjoub. Finding a $\tilde{G}_k$ and F_k with the correct structure ap-

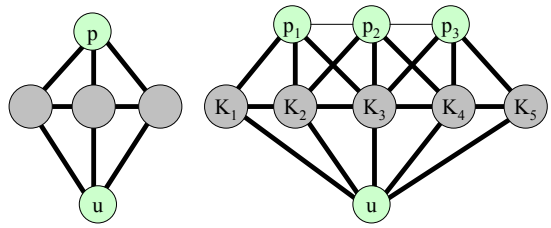

Figure 5: Diamonds with odd size.

pears to be difficult. A subtle requirement is that $\tilde{G}_k[\tilde{C}]$ has to have exactly $|\tilde{C}|$ affinely independent maximum stable sets. Otherwise, the matrix A is not invertible and Lemma 4 fails. Without this restriction, Theorem 1 would have held for any cut-set which partitions into two cliques.

There exist many graphs $\tilde{G}_k[\tilde{C}]$ with exactly $|\tilde{C}|$ affinely independent maximum stable sets, but the inequalities which define F_k must also involve the w_i vertices in a structured way. This structure would most likely involve extending the results of Section 2 to prevent the projection step from becoming too complicated.

6 Acknowledgements

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