Degree-Bounded Vertex Partitions

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Abstract

This paper studies degree-bounded vertex partitions. In particular, it discusses analogues for well-known results on the chromatic number and graph perfection.

1 Preliminaries

All graphs $G = (V, E)$ in this paper are finite, undirected, and simple. Given a vertex $v \in V$, define $N_G(v) := \{u \in V \mid uv \in E\}$, $\text{deg}_G(v) := |N_G(v)|$, $\Delta(G) := \max_{v \in V} \text{deg}_G(v)$, and $\delta(G) := \min_{v \in V} \text{deg}_G(v)$. The complement graph $\overline{G} = (V, \overline{E})$ has edge set $\overline{E}$, where $e \in \overline{E} \iff e \notin E$. The girth $g(G)$ is the length of a smallest cycle in $G$. For a subset $K \subseteq V$, $G[K]$ denotes the subgraph induced by $K$. In this paper, $k \geq 1$ is always a positive integer.

Definition 1. $K \subseteq V$ induces a $k$-plex if $\delta(G[K]) \geq |K| - k$.

Definition 2. $S \subseteq V$ induces a co-$k$-plex if $\Delta(G[S]) \leq k - 1$.

Definition 3. A partition of the vertex set into disjoint, nonempty co-$k$-plexes defines a co-$k$-plex coloring of $G$.

Seidman and Foster [5] introduced Definitions 1 and 2 and showed that a co-$k$-plex in $G$ is a $k$-plex in $\overline{G}$. Let $\omega_k(G)$ denote the cardinality of a largest $k$-plex in $G$. Notice that 1-plexes are complete subgraphs. Let $\alpha_k(G)$ denote the cardinality of a largest co-$k$-plex in $G$. Notice that co-1-plexes consist of pairwise nonadjacent vertices, or stable sets. The terms $k$-plex and co-$k$-plex refer to both the vertex sets and the corresponding induced subgraphs. This paper studies co-$k$-plex colorings and their relationship with $\omega_k(G)$.

A coloring of $G$ is a function $c_m : V \mapsto \{1, ..., m\}$ such that $c_m(u) \neq c_m(v)$ for each edge $uv \in E$. The chromatic number, $\chi(G)$, of $G$ is the smallest $m$ for which there exists a coloring $c_m$. If $K \subseteq V$ induces a 1-plex, then $c_m(u) \neq c_m(v)$ for all $u, v \in K$. It follows that $\omega_1(G) \leq \chi(G)$.

A coloring function partitions $V$ into co-1-plexes to obtain an upper bound on $\omega_1(G)$. Similarly, partitioning $V$ into degree-bounded subgraphs leads to an upper bound on $\omega_k(G)$. Let $S_1, ..., S_m$ be a co-$k$-plex coloring of $G$, and let $K \subseteq V$ be a maximum $k$-plex in $G$. Observe that

$$\omega_k(G) = |K| = \sum_{i=1}^{m} |K \cap S_i| \leq \sum_{i=1}^{m} \omega_k(G[S_i]),$$

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where the inequality follows from the fact that k-plexes are closed under set inclusion [5]. Let $\Pi$ denote the set of all co-k-plex colorings of $G$.

**Definition 4.** The co-k-plex chromatic number of $G$ is defined as

$$\chi_k(G) := \min \left\{ \sum_{S \in P} \omega_k(G[S]) : P \in \Pi \right\}.$$ 

Notice that $\chi_k(G) \geq \omega_k(G)$. Moreover, $\chi_1(G) = \chi(G)$ since $\omega_1(S) = 1$ for any co-1-plex $S$. An **optimal** co-k-plex coloring $S_1, ..., S_m$ satisfies $\chi_k(G) = \sum_{i=1}^m \omega_k(G[S_i])$. Section 2 uses the following result:

**Lemma 1.** If $G$ has at least $k$ vertices, then there exists an optimal co-k-plex coloring $S_1, ..., S_m$ of $G$ such that $|S_j| \geq k$ for some $j$.

**Proof.** Suppose the lemma is false. Choose an optimal coloring $S_1, ..., S_m$ with $|S_1|$ maximum. Notice that $m \geq 2$ since $|V| \geq k$ and $|S_i| < k$ for all $i$. Moreover, $|S_i| < k$ implies that $\omega_k(G[S_i]) = |S_i|$. Choose $v \in S_2$. Define $S'_1 := S_1 \cup \{v\}$ and $S'_2 := S_2 \setminus \{v\}$. Notice that

$$\chi_k(G) = \sum_{i=1}^m \omega_k(G[S_i]) = \sum_{i=1}^m |S_i| = |S'_1| + |S'_2| + \sum_{i=3}^m |S_i|,$$

so $S'_1, S'_2, ..., S_m$ is an optimal co-k-plex coloring such that $|S'_j| > |S_1|$. This contradicts the maximality of $S_1$. \qed

Section 2 defines k-plex perfection, contains examples of k-plex perfect graphs, and explores k-plex analogues for certain properties of perfect graphs. Section 3 establishes the existence graphs with large girth and large co-k-plex chromatic number. This result is a straightforward generalization of a famous theorem by Erdős.

## 2 Perfection

This section develops and studies a notion of k-plex perfection. Recall that a graph $G$ is **perfect** if $\chi(G') = \omega_1(G')$ for every vertex-induced subgraph $G' \subseteq G$.

**Definition 5.** A k-plex perfect graph $G$ satisfies $\omega_k(G') = \chi_k(G')$ for all vertex-induced subgraphs $G' \subseteq G$.

It is clear that any co-k-plex $S$ is k-plex perfect since $\chi_k(S) = \omega_k(S)$ by definition. Therefore, k-plex perfection follows from the fact that every vertex-induced subgraph of a co-k-plex is also a co-k-plex [5]. Recall that a finite set $X$ and a family $\mathcal{I}$ of subsets of $X$ define a matroid if the following axioms hold:

1. $\emptyset \in \mathcal{I}$
2. $I' \subseteq I \in \mathcal{I}$ implies $I' \in \mathcal{I}$
3. Every maximal set in $\mathcal{I}$ has the same cardinality
Given a graph \( G = (V, E) \), define
\[
\mathcal{K} = \{K \subseteq V : \delta(G[K]) \geq |K| - k\}.
\]
\( \mathcal{K} \) is the set of \( k \)-plexes in \( G \), and \((V, \mathcal{K})\) satisfies the first two matroid axioms for any graph.

**Theorem 1.** If \( M := (V, \mathcal{K}) \) defines a matroid, then \( G \) is \( k \)-plex perfect.

**Proof.** Given any vertex-induced subgraph \( G' = (V', E') \), define \( D := V \setminus V' \) and \( \mathcal{K}' = \{K \subseteq V' : \delta(G[K]) \geq |K| - k\} \). Observe that
\[
(V', \mathcal{K}') = (V \setminus D, \mathcal{K}') =: M \setminus D
\]
is again a matroid known as a deletion matroid, so it suffices to show \( \chi_k(G) = \omega_k(G) \).

Define \( x(A) = \sum_{a \in A} x_a, S = \{S \subseteq V : \Delta_i(G[S]) \leq k - 1\} \), and \( S_v = \{S \in S : v \in S\} \). Consider the following dual pair of linear programs:
\[
\text{max} \left\{ x(V) : x \geq 0, x(S) \leq \omega_k(G[S]) \text{ for all } S \in S \right\} \\
\text{min} \left\{ \sum_{S \in S} \omega_k(G[S])y_S : y \geq 0, y(S_v) \geq 1 \text{ for all } v \in V \right\}. \tag{1} \tag{2}
\]

Since \( M \) is a matroid, a theorem of Edmonds [2] implies that optimal solutions for (1) and (2) are integral. Observe that \( \omega_k(G) \) and \( \chi_k(G) \) are the optimal objective values for (1) and (2), respectively. Moreover, \( \omega_k(G) = \chi_k(G) \) by strong duality. \( \square \)

**Corollary 1.** If \( G \) is a \( k \)-plex, then \( G \) is \( k \)-plex perfect.

**Proof.** Given any \( K' \subset V \) and \( v \in V \setminus K' \), \( K' \cup \{v\} \) defines a \( k \)-plex. It follows that all maximal \( k \)-plexes have cardinality \( \omega_k(G) = |V| \), so \( G \) is \( k \)-plex perfect by Theorem 1. \( \square \)

Recall that an \( r \)-partite graph is \( r \)-colorable. The complete \( r \)-partite graphs have all possible edges between distinct color classes.

**Theorem 2.** If \( G \) is the complete \( r \)-partite graph \( K_{n_1, \ldots, n_r} \), then \( G \) is \( k \)-plex perfect.

**Proof.** Let \( K \) be a maximal \( k \)-plex in \( G \) and \( S_i \) the \( i \)-th partition class. Clearly, \( |K \cap S_i| \leq |S_i| = n_i \). In addition, \( |K \cap S_i| \leq k \). For if not, let \( v \in K \cap S_i \), and notice that \( N_G(v) \cap S_i = \emptyset \) implies
\[
\text{deg}_{G[K]}(v) = |K| - |K \cap S_i| < |K| - k,
\]
which contradicts that \( K \) is a \( k \)-plex. Therefore, \( |K \cap S_i| \leq \min\{k, n_i\} \) for each \( S_i \).

Suppose for contradiction that \( |K| = \sum_{i=1}^r |K \cap S_i| < \sum_{i=1}^r \min\{k, n_i\} \). Then there exists a \( j \) such that \( |K \cap S_j| < \min\{k, n_j\} \), and \( |K \cap S_j| < n_j \) implies that there exists a vertex \( v \in S_j \setminus K \). Consider the set \( K' := K \cup \{v\} \) and a vertex \( u \in K' \setminus S_j \). Since \( uv \in E \),
\[
\text{deg}_{G[K']} (u) = \text{deg}_{G[K]}(u) + 1 \geq (|K| - k) + 1 = |K'| - k.
\]
Now suppose \( u \in K \cap S_j \). Observe that \( \text{deg}_{G[K']} (u) = \text{deg}_{G[K]}(u) = |K| - |K \cap S_j| > |K| - k \) since \( uv \notin E \) and \( |K \cap S_j| < k \). It follows that
\[
\text{deg}_{G[K']} (u) \geq |K| - k + 1 = |K'| - k.
\]
Thus, since \( \text{deg}_{G[K']} (u) = \text{deg}_{G[K']} (v) \), \( K' \) is a \( k \)-plex in \( G \), which contradicts the maximality of \( K \). It follows that all maximal \( k \)-plexes in \( G \) have cardinality \( \sum_{i=1}^r \min\{k, n_i\} \), so \( G \) is \( k \)-plex perfect by Theorem 1. \( \square \)
It turns out that many properties of perfect graphs do not have k-plex analogues. Consider the complement $\overline{K}_{r,r}$ of a complete bipartite graph. Both components $H_1$ and $H_2$ of $\overline{K}_{r,r}$ are complete subgraphs.

**Lemma 2.** Let $k \geq 1$. If $r = 2k - 1$, then $\alpha_k(\overline{K}_{r,r}) = 2k$ and $\omega_k(\overline{K}_{r,r}) = 2k - 1$.

**Proof.** In the proof of Theorem 2, it was shown that

$$\omega_k(\overline{K}_{r,r}) = \sum_{i=1}^{2} \min\{k, r\} = 2k.$$ 

Thus, $\alpha_k(\overline{K}_{r,r}) = \omega_k(\overline{K}_{r,r}) = 2k$.

Now $\omega_k(\overline{K}_{r,r}) \geq 2k - 1$ because each component $H_i$ is complete and hence a $k$-plex of cardinality $2k - 1$. Suppose for contradiction that $\omega_k(\overline{K}_{r,r}) > 2k - 1$. Then there exists a $k$-plex $K \subseteq V$ such that $|K| = 2k$. If $|K \cap H_i| \leq k$, then

$$\text{deg}_{\overline{K}_{r,r}[K]}(v) \leq k - 1 < k = |K| - k \quad \text{for all} \quad v \in K \cap H_i.$$ 

This contradicts the definition of $k$-plex. Therefore, $|K \cap H_1| > k$ and $|K \cap H_2| > k$, which contradicts $|K| = 2k$. \hfill \square

**Theorem 3.** Let $k > 1$. If $r = 2k - 1$, then $\overline{K}_{r,r}$ is not $k$-plex perfect.

**Proof.** By Lemma 2, it suffices to show that $\chi_k(\overline{K}_{r,r}) \geq 2k$. Clearly, $\chi_k(\overline{K}_{r,r}) \geq \omega_k(\overline{K}_{r,r}) = 2k - 1$. Suppose for contradiction that $\chi_k(\overline{K}_{r,r}) = 2k - 1$. Lemma 1 implies the existence of an optimal co-$k$-plex coloring $S_1, ..., S_m$ of $\overline{K}_{r,r}$ such that $|S_1| \geq k$. Therefore, $\omega_k(\overline{K}_{r,r}[S_1]) \geq k$. Furthermore, $\chi_k(\overline{K}_{r,r}) < 2k$ implies that all other sets $S_i$ satisfy $|S_i| < k$. Notice that

$$2k - 1 = \chi_k(\overline{K}_{r,r}) = \sum_{i=1}^{m} \omega_k(\overline{K}_{r,r}[S_i]) \geq k + \sum_{i=2}^{m} \omega_k(\overline{K}_{r,r}[S_i]) = k + \sum_{i=2}^{m} |S_i|.$$ 

Consequently, $k - 1 \geq \sum_{i=2}^{m} |S_i|$. Now since the sets $S_i$ partition $V$ and $|V| = 4k - 2$,

$$|S_1| = |V| - \sum_{i=2}^{m} |S_i| \geq 3k - 1.$$ 

Therefore, $k > 1$ implies that $|S_1| \geq 3k - 1 > 2k$. This contradicts Lemma 2 because $S_1$ is a co-$k$-plex and $\alpha_k(\overline{K}_{r,r}) = 2k$. \hfill \square

Lovász’s [4] replication lemma is a well-known result from the theory of perfect graphs. Replication of a vertex $v \in V$ corresponds to the following operation: create a new vertex $v'$ and join it to $v$ and all the neighbors of $v$. The replication lemma states that replication of a vertex in a perfect graph produces another perfect graph. However, for $k \geq 2$, replication of a vertex in a $k$-plex perfect graph does not necessarily produce another $k$-plex perfect graph.

Fix $k > 1$. Consider the edgeless graph $G$ on two vertices $v_1$ and $v_2$. $G$ is a co-$k$-plex since $\Delta(G) = 0$. It follows that $G$ is $k$-plex perfect. Construct $G'$ by performing $2k - 2$ replication operations on each of $v_1$ and $v_2$. This construction implies that $G' = \overline{K}_{r,r}$, which
is not $k$-plex perfect by Theorem 3. Therefore, vertex replication does not preserve $k$-plex perfection.

Theorem 3 also illustrates the following interesting property: $G$ might not be $k$-plex perfect even if all components of $G$ are $k$-plex perfect. This statement follows from Corollary 1 and Theorem 3.

The final topic of this section is a $k$-plex version of the Weak Perfect Graph Theorem [4]. The Weak Perfect Graph Theorem states that $G$ is perfect if and only if $\overline{G}$ is perfect. Theorems 2 and 3 together provide counterexamples for $k$-plex analogues of the Weak Perfect Graph Theorem for any $k \geq 2$.

3 Girth and Co-$k$-plex Coloring

In 1959, Erdős [3] showed that the difference $\chi_k(G) - \omega_k(G)$ can be arbitrarily large. More precisely, he showed that for every integer $r \geq 1$, there exists a graph $G'$ with girth $g(G') > r$ and chromatic number $\chi(G') > r$. Observe that $g(G) > 3$ implies $\omega_1(G) \leq 2$. Therefore, Erdős’s theorem establishes the existence of graphs with high chromatic number and low clique number. Analogously, one might ask if the difference $\chi_k(G) - \omega_k(G)$ can also be arbitrarily large. This section shows that Erdős’s theorem does indeed generalize.

Lemma 3. Every co-$k$-plex $S$ has at most $\frac{|S|(k-1)}{2}$ edges.

Proof. $|E(S)| = \frac{1}{2} \cdot \sum_{v \in V(S)} \deg_{G[S]}(v) \leq \frac{1}{2} \cdot \sum_{v \in V(S)} (k-1) = \frac{|S|(k-1)}{2}$, where the inequality follows from the definition of co-$k$-plex. \hfill $\Box$

The following lemma and theorem together offer a straightforward generalization of the Erdős theorem. The proofs have been adapted from [1]. Let $G_{n,p}$ be the random graph on $n$ vertices where each edge exists with probability $0 \leq p \leq 1$. Let $q = 1 - p$.

Lemma 4. For all integers $n \geq t \geq k + 1$, the probability that $G \in G_{n,p}$ has a co-$k$-plex of size $t$ is at most

$$P[\alpha_k(G) \geq t] \leq \binom{n}{t} q^{(t-k)/2}.$$

Proof. Consider a fixed $t$-set $U \subseteq V$. By Lemma 3, the event that $U$ is a co-$k$-plex is contained in the event that

$$|E(G[U])| \leq \frac{t(k-1)}{2}. \quad (3)$$

Thus, the probability of the latter is an upper bound on the probability of the former. Now (3) requires that at least $\binom{t}{2} - \frac{t(k-1)}{2}$ edges are missing. That is, (3) occurs with probability at most $q^{\binom{t}{2} - \frac{t(k-1)}{2}}$. The lemma follows from the fact that $G$ contains $\binom{n}{t}$ $t$-sets. \hfill $\Box$

It is worth mentioning that if $t \leq \min\{k, n\}$, then every $t$-set is a co-$k$-plex, and Lemma 4 fails.

Theorem 4. Given any integer $r > k$, there exists a graph $G$ with girth $g(G) > r$ and co-$k$-plex chromatic number $\chi_k(G) > r$. 

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Proof. For $n$ large, suppose $t \geq \frac{n}{2r} > k$ and $(8r \ln n)n^{-1} \leq p \leq 1$. By Lemma 4,

$$P[\alpha_k \geq t] \leq \left( \frac{n}{t} \right) q^{t(t-k)/2} \leq n' q^{t(t-k)/2} = (nq^{(t-k)/2})^t \leq (ne^{-p(t-k)/2})^t.$$ 

Therefore,

$$P[\alpha_k \geq t] \leq (ne^{-pt/2}e^{pk/2})^t \leq (ne^{-2(\ln n)}e^{k/2})^t = (n^{-1}e^{k/2})^t.$$ 

Now $\lim_{n \to \infty} n^{-1}e^{k/2} = 0$, so

$$P[\alpha_k \geq \frac{n}{2r}] < \frac{1}{2}$$ (4)

for sufficiently large $n$.

On the other hand, fix $\epsilon$ with $0 < \epsilon < 1/r$, and let $X(G)$ denote the number of cycles of length at most $r$ in $G \in \mathcal{G}_{n,p}$. Erdős showed (see [1]) that for large $n$ and $p = n^{\epsilon-1}$,

$$P[X \geq \frac{n}{2}] < \frac{1}{2}.$$ (5)

Finally, fix $n$ large enough to satisfy (4), (5), and $n^{\epsilon-1} \geq (8r \ln n)n^{-1}$. Let $p = n^{\epsilon-1}$. There exists a $G \in \mathcal{G}_{n,p}$ such that $\alpha_k(G) < \frac{n}{2r}$ and $G$ has less than $\frac{n}{2}$ cycles of length at most $r$. Construct the graph $H$ by removing a vertex from each cycle of length at most $r$. Then $|H| \geq \frac{n}{2}$ and $g(H) > r$. Furthermore, $\alpha_k(H) \leq \alpha_k(G) < \frac{n}{2r}$ implies that any co-$k$-plex coloring of $H$ requires more than $r$ co-$k$-plex sets. Consequently, $\chi_k(H) > r$. □

Corollary 2. Given any integer $r > k + 2$, there exists a graph $G$ with $\chi_k(G) > r$ and $\omega_k(G) < k + 2$.

Proof. If $\omega_k(G) \geq k + 2$, then $G$ contains a $k$-plex $K$ of cardinality $k + 2$. Moreover, $\delta(G[K]) \geq 2$ by definition of $k$-plex. It follows that $G[K] \subseteq G$ contains a cycle of length at most $k + 2 = |K|$. Therefore, $g(G) > k + 2$ implies that $\omega_k(G) < k + 2$. The assertion now follows from Theorem 4. □

References


