A Note on Integer Domination of Cartesian Product Graphs

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Abstract

Given a graph G, a dominating set D is a set of vertices such that any vertex in G has at least one neighbor (or possibly itself) in D. A $\{k\}$-dominating multiset D_k is a multiset of vertices such that any vertex in G has at least k vertices from its closed neighborhood in D_k when counted with multiplicity. In this paper, we utilize the approach developed by Clark and Suen (2000) and properties of binary matrices to prove a "Vizing-like" inequality on minimum $\{k\}$-dominating multisets of graphs G, H and the Cartesian product graph $G\Box H$. Specifically, denoting the size of a minimum $\{k\}$-dominating multiset as $\gamma_{\{k\}}(G)$, we demonstrate that $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\ \gamma_{\{k\}}(G\Box H)$.

1 Introduction

Let G be a simple undirected graph $G = (V, E)$ with vertex set V and edge set E. The open neighborhood of a vertex $v \in V(G)$ is denoted by $N_G(v)$, and the closed neighborhood of v is denoted by $N_G[v]$. A dominating set D of a graph G is a subset of $V(G)$ such that for all $v \in V(G)$, $N_G[v] \cap D \neq \emptyset$, and the size of a minimum dominating set is denoted by $\gamma(G)$. The Cartesian product of two graphs G and H, denoted $G\Box H$, is the graph with vertex set $V(G) \times V(H)$, where vertices $gh, g'h' \in V(G\Box H)$ are adjacent whenever $g = g'$ and $(h, h') \in E(H)$, or $h = h'$ and $(g, g') \in E(G)$ (see Example 1).

In 1963, and again more formally in 1968, V. Vizing proposed a simple and elegant conjecture that has subsequently become one of the most famous open questions in domination theory.

Conjecture (Vizing [11], 1968)**.** *Given graphs G and H, $\gamma(G)\gamma(H) \leq \gamma(G\Box H)$.*

Although easy to state, a definitive proof of Vizing's conjecture (or a counter-example) has remained elusive. Over the past forty years (see [1] and references therein), Vizing's conjecture has been shown to hold on certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have been gradually tightened. Additionally, as numerous direct attempts on the conjecture have failed, research approaches expanded to include explorations of similar inequalities for total, paired, and fractional domination [6]. However, the most significant breakthrough occurred in 2000, when Clark and Suen [4] demonstrated that $\gamma(G)\gamma(H) \leq 2\gamma(G\Box H)$. This "Vizing-like" inequality immediately suggested similar inequalities for total [8] and paired [9] domination (2008 and 2010, respectively). In 2011, Choudhary, Margulies and Hicks [3] improved the inequalities from [8, 9] for total and paired domination by applying techniques similar to those of Clark and Suen, and also specific properties of binary matrices. In this paper, we explore *integer domination* (or $\{k\}$-domination), and again generate an improved inequality with this combined technique.

A *multiset* is a set in which elements are allowed to appear more than once, e.g. $\{1, 2, 2\}$. All graphs and multisets in this paper are finite. A $\{k\}$-dominating multiset D_k of a graph G is a multiset of vertices of $V(G)$ such that, for each $v \in V(G)$, the number of vertices of $N_G[v]$ contained in D_k (counted with multiplicity) is at least k. A $\gamma_{\{k\}}$-set of G is a minimum $\{k\}$-dominating multiset, and the size of a minimum $\{k\}$-dominating multiset is denoted by $\gamma_{\{k\}}(G)$. Additionally, note that a $\{1\}$-dominating multiset is equivalent to the standard dominating set.

The notion of a $\{k\}$-dominating *multiset* is equivalent to the more familiar notion of a $\{k\}$-dominating *function*. The study of $\{k\}$-dominating functions was first introduced by Domke, Hedetniemi, Laskar, and Fricke [5] (see also [7], pg. 90), and further explored by Brešar, Henning and Klavžar in [2]. In [10], the authors investigate integer domination in terms of graphs with specific packing numbers, and in [2], various applications of $\{k\}$-dominating functions are described, such as physical locations (stores, buildings, etc.) serviced by up to k fire stations (as opposed to the single required fire station in the canonical application of dominating sets). Finally, in [2], the authors prove the following "Vizing"-like inequality:

Theorem ([2])**.** *Given graphs G and H, $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq k(k+1)\gamma_{\{k\}}(G\Box H)$.*

Observe that for $k = 1$, the Brešar-Henning-Klavžar theorem is equivalent to the bound proven by Clark and Suen. In this paper, we improve this upper bound from $O(k^2)$ to $O(k)$, and prove the following theorem:

Theorem. *Given graphs G and H, $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\ \gamma_{\{k\}}(G\Box H)$.*

Again, observe that for $k = 1$, this theorem is equivalent to the bound proven by Clark and Suen. In the next section, we develop the necessary background and present the proof.

2 Background, Notation and Proof of Theorem

In this section, we introduce the necessary background and notation used throughout the paper, and prove several propositions to streamline the proof of Theorem 1.

For $gh \in V(G\Box H)$, the G-neighborhood $\big($denoted by $N_{\underline{\mathbf{G}}\Box H}(gh)\big)$ and the H-neighborhood $\big($denoted by $N_{G\Box\underline{\mathbf{H}}}(gh)\big)$ are defined as follows:

$$N_{\underline{\mathbf{G}}\Box H}(gh) = \{g'h \in V(G\Box H) \mid g' \in N_G(g)\} \ ,$$
$$N_{G\Box\underline{\mathbf{H}}}(gh) = \{gh' \in V(G\Box H) \mid h' \in N_H(h)\} \ .$$

Thus, $N_{\underline{\mathbf{G}}\square H}(gh)$ and $N_{G\square\underline{\mathbf{H}}}(gh)$ are both subsets of $V(G\square H)$. Additionally, the edge set $E(G\square H)$ can be partitioned into two sets (**G**-edges and **H**-edges) where

$$\mathbf{G}\text{-edges} = \{(gh, g'h) \in E(G\square H) \mid h \in V(H) \text{ and } (g, g') \in E(G)\} \ ,$$
$$\mathbf{H}\text{-edges} = \{(gh, gh') \in E(G\square H) \mid g \in V(G) \text{ and } (h, h') \in E(H)\} \ .$$

Given a dominating set D of a graph G and a vertex $v \in V(G)$, we say that the vertices in $N_G[v] \cap D$ are the *v-dominators* in D.

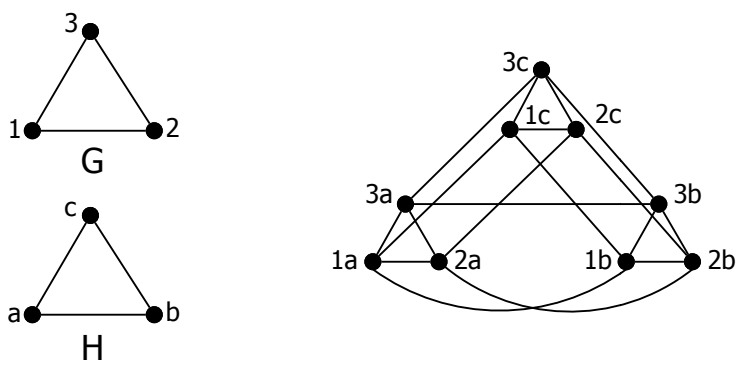
A *union of multisets* is denoted by $\uplus$, e.g. $\{1,2,2\} \uplus \{1,2,3\} = \{1,1,2,2,2,3\}$. The union of a multiset with itself t times is denoted by $\uplus^t$, e.g. $\uplus^2\{1,2,2\} = \{1,2,2\} \uplus \{1,2,2\} = \{1,1,2,2,2,2\}$. The *cardinality of a multiset* is equal to the summation over the number of occurrences of each of its elements, e.g. $\big|\{1,2,2\}\big| = 3$, and given a multiset A, we denote the number of occurrences of a particular element a in A as $|A|_a$, e.g. $\big|\{1,2,2\}\big|_2 = 2$, and $\big|\{1,2,2\}\big|_4 = 0$. A multiset B is a *sub-multiset* of multiset A if each element $b \in B$ is present in A, and $|B|_b \leq |A|_b$, e.g. $\{1,2,2\} \subseteq \{1,2,2,2\} \subsetneq \{1,2\}$. Finally, let A be a multiset and B a set. Then, $|A|_B = \sum_{b\in B} |A|_b$. For example, $\{1,1,2,5,6,6\}_{\{1,4,6\}} = 4$.

Given graphs G and H, let $A \subseteq \uplus^t V(G\square H)$, where t is any positive integer. When defining a multiset, we must not only describe the elements contained in the multiset, but also define the number of times a specific element appears in the multiset. Thus, the Φ-projection and Ψ-projection of A on graphs G and H are multisets defined as

$$\Phi_G(A) = \Big\{g \in V(G) \ : \ \exists\, h \in V(H) \text{ with } gh \in A, \text{ where } \big|\Phi_G(A)\big|_g = \max\big\{|A|_{gh} \ : \ h \in V(H)\big\}\Big\} \ ,$$
$$\Phi_H(A) = \Big\{h \in V(H) \ : \ \exists\, g \in V(G) \text{ with } gh \in A, \text{ where } \big|\Phi_H(A)\big|_h = \max\big\{|A|_{gh} \ : \ g \in V(G)\big\}\Big\} \ ,$$
$$\Psi_G(A) = \Big\{g \in V(G) \ : \ \exists\, h \in V(H) \text{ with } gh \in A, \text{ where } \big|\Psi_G(A)\big|_g = \textstyle\sum_{h\in V(H)} |A|_{gh}\Big\} \ ,$$
$$\Psi_H(A) = \Big\{h \in V(H) \ : \ \exists\, g \in V(G) \text{ with } gh \in A, \text{ where } \big|\Psi_H(A)\big|_h = \textstyle\sum_{g\in V(G)} |A|_{gh}\Big\} \ .$$

Note that multisets $\Phi_G(A)$ and $\Psi_G(A)$ contain identical elements, but the number of occurrences of a given g in $\Phi_G(A)$ is defined by a *max*, whereas the number of occurrences of the same g in $\Psi_G(A)$ is defined by a *sum*. This *max/sum* distinction in these multiset definitions will play a critical role of our proof of Theorem 1. We now present an example of $\Phi_G(A)$ and $\Psi_G(A)$.

Example 1. *Consider graphs G, H and $G\square H$:*

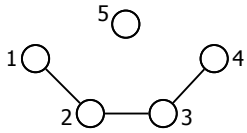


(a) G, H and $G\square H$

Let $A = \{1b, 1c, 1c, 2a, 2a, 2a, 2b\} \subseteq \uplus^3 V(G\square H)$. Note $|A|_{1b} = 1, |A|_{1c} = 2, |A|_{2a} = 3$, and $|A|_{2b} = 1$. Then $\max\big\{|A|_{1h} : h \in V(H)\big\} = 2, \max\big\{|A|_{2h} : h \in V(H)\big\} = 3, \sum_{h\in V(H)} |A|_{1h} = 3$ and $\sum_{h\in V(H)} |A|_{2h} = 4$. Therefore, $\Phi_G(A) = \{1,1,2,2,2\}$ and $\Psi_G(A) = \{1,1,1,2,2,2,2\}$. $\square$

Let $\{P_1, P_2, \ldots, P_t\}$ be a multiset of subsets of a set A. Then $P^A = \{P_1, P_2, \ldots, P_t\}$ is a *k-partition* of A if each element of A is present in exactly k of the sets $P_1, \ldots, P_t$. For example, given $A = \{1, 2, 3, 4, 5, 6, 7\}$, then $P^A = \big\{\{1, 2, 3\}, \{1, 2, 3\}, \{4, 5, 6\}, \{6, 7\}, \{5, 7\}, \{4\}\big\}$ is a 2-partition of A. We observe that the subset $\{1, 2, 3\}$ is present twice in P^A, demonstrating that a k-partition can be a multiset.

Example 2. *Consider the following graph G:*



Here, a 2-partition of $V(G)$ is $P^G = \big\{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{4\}, \{5\}, \{5\}\big\} = \{P_1^G, \ldots, P_6^G\}$. Observe that each $v \in V(G)$ appears in exactly two *sets in P^G. Also, note that each P_i^G is a* set *(i.e., it contains no duplicated elements), but P^G itself is a* multiset. *Finally, observe that a minimum 2-dominating multiset of G is $\{1, 2, 3, 4, 5, 5\}$, and thus $\gamma_{\{2\}}(G) = 6$.* $\square$

Given a graph G, we will now define the concept of domination among multisets of $V(G)$. Given a positive integer t and multisets $A, B \subseteq \uplus^t V(G)$, we say that A *dominates* B if, for each $b \in B$, the number of vertices of $N_G[b]$ present in A (counted with multiplicity) is *at least* the number of occurrences of $b \in B$. In other words,

$$|A|_{N_G[b]} \geq |B|_b \ , \qquad \text{for each } b \in B \ .$$

The following proposition can now be verified.

Proposition 1. *Given graphs G, H, and positive integers t, k, with $t \geq k$, then*

1. *A multiset $A \subseteq \uplus^t V(G)$ is a $\{k\}$-dominating multiset for G if and only if A dominates $\uplus^k V(G)$.*

2. *Given multisets $A, B, A', B' \subseteq \uplus^t V(G)$, if A dominates A', and B dominates B', then $A \uplus B$ dominates $A' \uplus B'$.*

3. *Given multisets $A, A' \subseteq \uplus^t V(G \square H)$, if A dominates A', then $\Psi_H(A)$ dominates $\Psi_H(A')$.*

The proof of Prop. 1 is a straight forward application of the definitions. Thus, we skip the proof for space considerations.

Proposition 2. *Given graphs G, H, let $\{u_1, \ldots, u_{\gamma_{\{k\}}(G)}\}$ and $\{\overline{u}_1, \ldots, \overline{u}_{\gamma_{\{k\}}(H)}\}$ be minimum $\{k\}$-dominating multisets of G, H, respectively, and let $P^G = \{P_1^G, \ldots, P^G_{\gamma_{\{k\}}(G)}\}$ and $P^H = \{P_1^H, \ldots, P^H_{\gamma_{\{k\}}(H)}\}$be a k-partitions of $V(G), V(H)$ respectively, such that $u_i \in P_i^G$ and $P_i^G \subseteq N_G[u_i]$ (and $\overline{u}_j \in P_j^H$ and $P_j^H \subseteq N_H[\overline{u}_j]$, respectively) for $1 \leq i \leq \gamma_{\{k\}}(G)$ and $1 \leq j \leq \gamma_{\{k\}}(H)$.*

1. *Let $I \subseteq \{1, \ldots, \gamma_{\{k\}}(G)\}$, $A = \uplus_{i \in I}\{u_i\}$, and $C = \uplus_{i \in I} P_i^G$. Then A dominates C. Furthermore, given an integer $k' \geq k$, let $B \subseteq \uplus^{k'} V(G)$ be any other multiset dominating C. Then $|B| \geq |A|$.*

2. *Let $J \subseteq \{1, \ldots, \gamma_{\{k\}}(H)\}$, $\overline{A} = \uplus_{j \in J}\{\overline{u}_j\}$, and $\overline{C} = \uplus_{j \in J} P_j^H$. Then $\overline{A}$ dominates $\overline{C}$. Furthermore, given an integer $k' \geq k$, let $\overline{B} \subseteq \uplus^{k'} V(H)$ be any other multiset dominating $\overline{C}$. Then $|\overline{B}| \geq |\overline{A}|$.*

Proof of Prop. 2.1: We will first prove A dominates C. Since $P_i^G \subseteq N_G[u_i]$, u_i dominates P_i^G. Therefore, $\uplus_{i\in I}\{u_i\}$ dominates $\uplus_{i\in I}P_i^G$, i.e. A dominates C. Now let multiset $W = \uplus_{i\notin I}\{u_i\}$. Since B is any multiset dominating C, $B\uplus W$ dominates $\left(\uplus_{i\in I} P_i^G\right)\uplus\left(\uplus_{i\notin I} P_i^G\right)$ (by Prop. 1.2). Since P^G is a k-partition, $\left(\uplus_{i\in I} P_i^G\right)\uplus\left(\uplus_{i\notin I} P_i^G\right) = \uplus^k V(G)$. Since $A\uplus W = \{u_1,\ldots,u_{\gamma_k(G)}\}$ is a $\gamma_{\{k\}}$-set of G, $A\uplus W$ dominates $\uplus^k V(G)$ (by Prop. 1.1). Therefore $|B\uplus W|\geq |A\uplus W|$, and $|B|\geq |A|$. The proof of Prop. 2.2 follows similarly. □

In the introduction, we stated that the proof of Theorem 1 relies on the double-projection technique of Clark and Suen, and also a particular property of binary matrices. Specifically, we have the following proposition:

Proposition 3. *Let M be a matrix containing only 0/1 entries. Then one (or both) of the following two statements are true:*

(a) each column contains a 1 ,

(b) each row contains a 0 .

Proof. For a proof by contradiction, assume there exists a row (say i) which does not contain a 0, and a column (say j) which does not contain a 1. Then, the entry $M[i,j]$ is neither 0 nor 1, which is a contradiction. □

We are now ready to state and prove the main theorem of the paper.

Theorem 1. *For graphs G and H, $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq 2k\ \gamma_{\{k\}}(G\Box H)$.*

Proof. We begin with the same notation as in Prop. 2. Let $\{u_1,\ldots,u_{\gamma_{\{k\}}(G)}\}$ and $\{\overline{u}_1,\ldots,\overline{u}_{\gamma_{\{k\}}(H)}\}$ be minimum $\{k\}$-dominating multisets of G, H, respectively, and let $P^G = \{P_1^G,\ldots,P^G_{\gamma_{\{k\}}(G)}\}$ and $P^H = \{P_1^H,\ldots,\ P^H_{\gamma_{\{k\}}(H)}\}$be a k-partitions of $V(G), V(H)$ respectively, such that $u_i \in P_i^G$ and $P_i^G \subseteq N_G[u_i]$ (and $\overline{u}_j \in P_j^H$ and $P_j^H \subseteq N_H[\overline{u}_j]$, respectively) for $1 \leq i \leq \gamma_{\{k\}}(G)$ and $1 \leq j \leq \gamma_{\{k\}}(H)$. Recall from Example 2 that P_i^G and P_j^G may be completely distinct, equal, or overlap in parts. Finally, observe that since P^G and P^H are k-partitions of $V(G), V(H)$, respectively, $P^G \times P^H$ is a k^2-partition of $V(G\Box H)$.

We now describe a notation for uniquely identifying *different* occurrences of the *same* vertex $gh \in V(G\Box H)$ in the k^2-partition $P^G \times P^H$. Let $I = \{1,\ldots,\gamma_{\{k\}}(G)\}$, $J = \{1,\ldots,\gamma_{\{k\}}(H)\}$, and $\overline{V} = \uplus^{k^2} V(G\Box H) = \uplus_{i\in I}\uplus_{j\in J}\left(P_i^G \times P_j^H\right)$. Since P^G is a k-partition of $V(G)$ and P^H is a k-partition of $V(H)$, for each $gh \in V(G\Box H)$, let $f_g : \{1,\ldots,k\} \to I$ and $f_h : \{1,\ldots,k\} \to J$ be one-to-one functions that identify the k blocks where vertex g appears in the k-partition P^G (and similarly for P^H). Thus, the k copies of g in P^G appear in blocks $P^G_{f_g(1)},\ldots,P^G_{f_g(k)}$, and similarly for P^H. Let $(gh)_{sr}$ (for $1 \leq s, r \leq k$) indicate the sr-th copy of vertex gh in the k^2-partition $\overline{V}$ that occurs due to block $P^G_{f_g(s)} \times P^H_{f_h(r)}$.

Let D_k be a minimum $\{k\}$-dominating multiset of $G\Box H$ and $gh \in V(G\Box H)$. Since D_k is a γ_k-set of $G\Box H$, there are *at least* k dominators $d_0,\ldots,d_{k-1}$ of gh in D_k (when counted with multiplicity, thus $d_0,\ldots,d_{k-1}$ are not necessarily distinct dominators). We now create a function to assign a specific dominator in D_k (not necessarily unique) to each of the k^2 copies of $gh \in \overline{V}$. Specifically, let $d : (gh)_{sr} \to \{d_0,\ldots,d_{k-1}\}$ (for $1 \leq s, r \leq k$) be defined as $d\big((gh)_{sr}\big) = d_{(s+r) \bmod k}$. Note that for s fixed and $1 \leq r \leq k$, the k copies of vertex $(gh)_{sr}$ are assigned dominators $\{d_{(s+1) \bmod k},\ldots,d_{(s+k) \bmod k}\} = \{d_0,\ldots,d_{k-1}\}$ in D_k (and similarly for r fixed).

We now define a binary matrix corresponding to each of $P_i^G \times P_j^H$ block (for $i \in I$, $j \in J$) based on the "type" of dominator assigned to a particular vertex gh. For $g \in P_i^G$ and $h \in P_j^H$, let $s = f_g^{-1}(i)$ and $r = f_h^{-1}(j)$. Then, we define the binary $|P_i^G| \times |P_j^H|$ matrix F_{ij} such that:

$$F_{ij}(g,h) = \begin{cases} 0 & \text{if } d\big((gh)_{sr}\big) \in N_{\underline{\mathbf{G}}\square H}(gh) \ , \\ 1 & \text{otherwise} \ . \end{cases}$$

Observe that for each $i \in I$ and $g \in P_i^G$, even though the function f_g is not onto, the inverse $f_g^{-1}(i)$ is always defined since one of the k copies of g in the k-partition P^G appears in block P_i^G (and similarly for $f_h^{-1}(j)$). Furthermore, observe that $F_{ij}(g,h) = 1$ in two cases: 1) if $d\big((gh)_{sr}\big)$ is vertex gh itself, or if $d\big((gh)_{sr}\big)$ dominates $(gh)_{sr}$ via an $\mathbf{H}$-edge.

By Prop. 3, each of the binary matrices F_{ij} satisfies one or both of the statements in Prop. 3. We will now define a series of multisets based on which of the properties F_{ij} satisfies.

$$\begin{aligned}
Z_i &= \big\{ \uplus\, (P_i^G \times P_j^H) : F_{ij} \text{ satisfies Prop. 3.a} \ , \ 1 \le j \le \gamma_{\{k\}}(H) \big\} \ , \quad \text{for } 1 \le i \le \gamma_{\{k\}}(G) \ , \\
\overline{Z}_j &= \big\{ \uplus\, (P_i^G \times P_j^H) : F_{ij} \text{ satisfies Prop. 3.b} \ , \ 1 \le i \le \gamma_{\{k\}}(G) \big\} \ , \quad \text{for } 1 \le j \le \gamma_{\{k\}}(H) \ , \\
N_i &= \big\{ \overline{u}_j : F_{ij} \text{ satisfies Prop. 3.a} \ , \ 1 \le j \le \gamma_{\{k\}}(H) \big\} \ , \quad \text{for } 1 \le i \le \gamma_{\{k\}}(G) \ , \\
\overline{N}_j &= \big\{ u_i : F_{ij} \text{ satisfies Prop. 3.b} \ , \ 1 \le i \le \gamma_{\{k\}}(G) \big\} \ , \quad \text{for } 1 \le j \le \gamma_{\{k\}}(H) \ , \\
Y_i &= \Phi_H(Z_i) = \big\{ \uplus\, P_j^H : F_{ij} \text{ satisfies Prop. 3.a} \ , \ 1 \le j \le \gamma_{\{k\}}(H) \big\} \ , \quad \text{for } 1 \le i \le \gamma_{\{k\}}(G) \ , \\
\overline{Y}_j &= \Phi_G(\overline{Z}_j) = \big\{ \uplus\, P_i^G : F_{ij} \text{ satisfies Prop. 3.b} \ , \ 1 \le i \le \gamma_{\{k\}}(G) \big\} \ , \quad \text{for } 1 \le j \le \gamma_{\{k\}}(H) \ , \\
S_i &= D_k \cap \Big(\uplus^k \ \big(P_i^G \times V(H)\big)\Big) \ , \quad \text{for } 1 \le i \le \gamma_{\{k\}}(G) \ , \\
\overline{S}_j &= D_k \cap \Big(\uplus^k \ \big(V(G) \times P_j^H\big)\Big) \ , \quad \text{for } 1 \le j \le \gamma_{\{k\}}(H) \ .
\end{aligned}$$

To clarify the definition of S_i, observe that the intersection of two multisets $\{1,1,1,2,2,3\} \cap \{1,1,2,4\} = \{1,1,2\}$. We will now prove the following claim.

Claim 1. *For $i = 1, \ldots, \gamma_{\{k\}}(G)$, $\Psi_H(S_i)$ dominates Y_i, and for $j = 1, \ldots, \gamma_{\{k\}}(H)$, $\Psi_G(\overline{S}_j)$ dominates $\overline{Y}_j$.*

Proof. In order to show that $\Psi_H(S_i)$ dominates Y_i, we must show that 1) every vertex $y \in Y_i$ is dominated by some vertex $h \in S_i$, and 2) the number of occurrences of y-dominators in the multiset $\Psi_H(S_i)$ is greater than or equal to the number of occurrences of y in multiset Y_i.

In order to prove (1), consider $y \in Y_i$. By definition, there exists a j such that $y \in P_j^H$ and F_{ij} satisfies Prop. 3.a. Since column $P_i^G \times y$ of F_{ij} contains a "1", there exists a $g \in P_i^G$ such that vertex gy is dominated by an $\mathbf{H}$-edge (or itself). Let gh be a dominator of gy. Thus, there exists an $h \in \Psi_H(S_i)$ such that h dominates y.

In order to prove (2), consider $y \in Y_i$. Let $|Y_i|_y = t$. Recall that $Y_i \subseteq \uplus_{j \in J} P_j^H = \uplus^k V(H)$. Since P^H is a k-partition of $V(H)$, y appears in exactly k blocks of P^H. Thus, $t \le k$.

Let $\{j_1, \ldots, j_t\}$ be such that matrices $F_{ij_1}, \ldots, F_{ij_t}$ satisfy Prop. 3.a, and y is contained in $P_{j_1}^H, \ldots, P_{j_t}^H$. Furthermore, let $g_1, \ldots, g_t \in P_i^G$ be such that $F_{ij_w}(g_w, y) = 1$ (for $1 \le w \le t$). Then, each of $g_w y$ is dominated by an $\mathbf{H}$-edge (or itself), and given $g_w \ne g_{w'}$ with $1 \le w, w' \le t$, vertices $g_w y$ and $g_{w'} y$ are dominated by distinct vertices in D_k (and by extension, S_i). However, we must now show that two identical vertices $g_w y, g_{w'} y$ (i.e., $w = w'$) have dominators with distinct indices in D_k.

Recall that different occurrences of vertex gh in the multiset $\overline{V}$ are denoted as $(gh)_{sr}$, indicating that the sr-th copy of gh is due to the $P_{f_g(s)}^G \times P_{f_h(r)}^H$ block. In this case, two identical

vertices $g_w y, g_{w'} y$ occur due to the same P_i^G block, but different P_j^H blocks. Furthermore, recall that the dominators $d_0, \ldots, d_{k-1}$ are assigned to the different occurrences of gh via the function $d\big((gh)_{sr}\big) = d_{(s+r) \bmod k}$, and that for s fixed and $1 \leq r \leq k$, the k copies of vertex $(gh)_{sr}$ are assigned dominators $\{d_{(s+1) \bmod k}, \ldots, d_{(s+k) \bmod k}\} = \{d_0, \ldots, d_{k-1}\} \subseteq D_k$. Therefore, the two identical vertices $g_w y$ and $g_{w'} y$ will map to dominators with distinct indices in D_k.

Finally, recall that the number of occurrences of a given $h \in \Psi_H(S_i)$ is determined due to the *sum* (as opposed to the *maximum*). Given a vertex $y \in Y_i$ where $|Y_i|_y = t$, we have demonstrated that there are *at least* t vertices in $S_i = D_k \cap \uplus^k \big(P_i^G \times V(H)\big)$ (when counted with multiplicity) whose projection on H dominates y, and therefore, there are at least t y-dominators appearing in $\Psi_H(S_i)$ (when counted with multiplicity).

To conclude, we have demonstrated that 1) every vertex $y \in Y_i$ is dominated by some vertex $h \in S_i$, and 2) the number of occurrences of y-dominators in the multiset $\Psi_H(S_i)$ is greater than or equal to the number of occurrences of y in multiset Y_i. Therefore, Y_i is dominated by $\Psi_H(S_i)$.

Similarly, we can demonstrate that $\overline{Y}_j$ is dominated by $\Psi_G(\overline{S}_j)$. □

We will now carefully bound the sizes of the sets $N_i, \overline{N}_j, S_i, \overline{S_j}$, etc., in relation to each other. We observe that the total number of $P_i^G \times P_j^H$ blocks in the k^2-partition of $V(G \square H)$ is $\gamma_{\{k\}}(G)\gamma_{\{k\}}(H)$. Since the binary matrix F_{ij} associated with each these blocks satisfies at least one of the two conditions of Prop. 3, we see that

$$\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq \sum_{i=1}^{\gamma_{\{k\}}(G)} |N_i| + \sum_{j=1}^{\gamma_{\{k\}}(H)} |\overline{N}_j| \ .$$

Since P_G is a k-partition of $V(G)$, every g appears in exactly k blocks of P^G. Thus, every $gh \in V(G \square H)$ appears in exactly k "strips" of $P^G \times V(H)$. Thus, if $gh \in \big(D_k \cap (P_i^G \times V(H)\big)$, then gh appears in "strip" $D_k \cap \big(\uplus^k P_i^G \times V(H)\big)$ exactly $|D_k|_{gh}$ times. Therefore, when we iterate over all the "strips", we see

$$\sum_{i=1}^{\gamma_{\{k\}}(G)} |S_i|_{gh} = k|D_k|_{gh} \ , \qquad \sum_{i=1}^{\gamma_{\{k\}}(G)} |S_i| = \sum_{j=1}^{\gamma_{\{k\}}(H)} |\overline{S}_j| = k|D_k| = k\gamma_{\{k\}}(G \square H) \ .$$

Since N_i dominates Y_i (Prop. 2.2), and since $\Psi_H(S_i)$ is another multiset dominating Y_i (Claim 1), we see by Prop. 2.2 that

$$|\Psi_H(S_i)| \geq |N_i| \ , \text{ for } 1 \leq i \leq \gamma_{\{k\}}(G) \ , \quad \text{and} \quad |\Psi_G(\overline{S}_j)| \geq |\overline{N}_j| \ , \text{ for } 1 \leq j \leq \gamma_{\{k\}}(H) \ .$$

Furthermore, since the number of occurrences of a given vertex in the Ψ-projection is determined by the *sum* (as opposed to the *maximum*), $|S_i| = |\Psi_H(S_i)|$, and $|\overline{S}_j| = |\Psi_G(\overline{S}_j)|$. Therefore,

$$|S_i| \geq |N_i| \ , \text{ for } 1 \leq i \leq \gamma_{\{k\}}(G) \ , \quad \text{and} \quad |\overline{S}_j| \geq |\overline{N}_j| \ , \text{ for } 1 \leq j \leq \gamma_{\{k\}}(H) \ .$$

Combining all of these inequalities together, we finally see

$$\gamma_{\{k\}}(G)\gamma_{\{k\}}(H) \leq \sum_{i=1}^{\gamma_{\{k\}}(G)} |N_i| + \sum_{j=1}^{\gamma_{\{k\}}(H)} |\overline{N}_j| \leq \sum_{i=1}^{\gamma_{\{k\}}(G)} |S_i| + \sum_{j=1}^{\gamma_{\{k\}}(H)} |\overline{S}_j| = 2k \ \gamma_{\{k\}}(G \square H) \ .$$

This concludes our proof. □

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