CO-2-PLEX POLYNOMIALS

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Abstract. This paper offers a generalization of the independence polynomial, the co-k-plex polynomial. The resulting family of polynomials carries combinatorial information on a class of independence systems defined over the vertex set of a finite graph. Specifically, we offer a recursion formula and examples of the co-2-plex polynomial on certain graphs. In addition, we characterize the class of graphs whose co-2-plex polynomial will have all real roots.

Key words. co-2-plex, independence polynomial

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1. Introduction. The graphs discussed in this paper are finite and simple. Refer to Diestel [6] for standard graph terminology. For a graph $G = (V, E)$ and $S \subseteq V$, let $G[S]$ be the subgraph induced by S. Given $v \in V$, define $N_G(v) = \{u \in V : vu \in E\}$, $N_G[v] = N_G(v) \cup \{v\}$, and $\Delta(G) = max_{v \in V} |N_G(v)|$. A set of pairwise nonadjacent vertices in G defines an *independent set*. Let $\mathcal{I}^G$ denote the set of all independent sets in G. Gutman and Harary [7] associated the following polynomial with G:

$$I(G; x) = \sum_{I \in \mathcal{I}^G} x^{|I|}.$$

This *independence polynomial* carries information about the enumerative structure of independent sets in G. More precisely, the coefficient of x^i in $I(G; x)$ is exactly the number of independent sets of cardinality i in G. The independence polynomial has been studied in a number of papers [1, 2, 3, 4, 5, 8, 9, 10, 11, 12, 13]. Levit and Mandrescu offer a survey [14].

Recall that a nonempty collection of subsets of V which is closed under set inclusion defines an *independence system*. Fix an integer $k \geq 1$ and let $S \subseteq V$ satisfy $|N_G[v] \cap S| \leq k \quad$ for all $v \in S$. The set S defines a *co-k-plex* in G. Let $\mathcal{I}_k^G$ denote the set of co-k-plexes in G. Notice that $\mathcal{I}_1^G = \mathcal{I}^G$ and that $\mathcal{I}_k^G$ defines an independence system on V for all integers $k \geq 1$. The graph G is associated with the family of *co-k-plex polynomials* defined as follows: $I_k(G; x) = \sum_{I \in \mathcal{I}_k^G} x^{|I|} \quad k = 1, 2, 3, ...$

Let s_i^k be the coefficient of x^i in $I_k(G; x)$; that is, s_i^k denotes the number of co-k-plexes of cardinality i in G. Clearly, $s_i^k = 0$ for all $i > \alpha_k(G)$, where $\alpha_k(G)$ denotes the size of a largest co-k-plex in G. Notice also that $S \in \mathcal{I}_k^G \Rightarrow S \in \mathcal{I}_{k+1}^G$. Consequently, $s_i^k \leq s_i^{k+1}$ for any k, and $I_k(G; x) = I_{k+1}(G; x)$ whenever $k > \Delta(G)$. In fact, $I_k(G; x) = (1 + x)^{|V(G)|}$ for all $k > \Delta(G)$ because, for this range of k, every subset of vertices defines a co-k-plex.

This paper is organized as follows. Section 2 explores the effect certain graph operations have on the corresponding polynomials and derive recursive relationships for the co-2-plex polynomial. In addition, Section 2.1 characterizes the class of graphs such that the corresponding co-2-plex polynomials have all real roots, using one of the aforementioned recursive relationships. Section 3 computes the co-2-plex polynomials

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for various structured graphs. Finally, Section 4 is reserved for conclusions and future work.

2. Graph Operations and Recursive Relationships.

This section investigates the effect certain graph operations have on the corresponding polynomials and derives recursive relationships for the co-2-plex polynomial. The first operation considered is graph union. The graph $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The graph $G = \bigcup_{i=1}^r G_i$ is defined inductively.

LEMMA 2.1. *Fix an integer $k \geq 1$. If $G = \bigcup_{i=1}^r G_i$, then $I_k(G;x) = \prod_{i=1}^r I_k(G_i;x)$.*

Proof. The result is trivial for $r = 1$, so suppose that $r = 2$. Notice that, given co-k-plexes $S_1 \subseteq G_1$ and $S_2 \subseteq G_2$, the set $S = S_1 \cup S_2$ is a co-k-plex in $G_1 \cup G_2$. Moreover, every co-k-plex in $G_1 \cup G_2$ can be constructed this way. It follows that the coefficient of x^i in the polynomial $I_k(G_1 \cup G_2; x)$ equals the sum of the product of all coefficients of pairs y^l in $I_k(G_1; y)$ and z^m in $I_k(G_2; z)$ such that $l + m = i$. In other words, $I_k(G_1 \cup G_2; x)$ is the product of $I_k(G_1; x)$ and $I_k(G_2; x)$. Now if $r > 2$, repeat this argument using graphs $\bigcup_{i=1}^{j-1} G_i$ and G_j for each $j = 3, ..., r$, □

The join of graphs G_1, G_2 is the graph $G = G_1 + G_2$, where $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$. It is well-known [1, 7, 10] that $I_1(G;x) = I_1(G_1;x) + I_1(G_2;x) - 1$. The following result generalizes this formula to the case where $k = 2$.

THEOREM 2.2. *Let G_1 and G_2 be graphs with n_1 and n_2 vertices, respectively. If $G = G_1 + G_2$, then*

$$I_2(G;x) = I_2(G_1;x) + I_2(G_2;x) + \sum_{j=0}^{2} \left[\binom{n_1+n_2}{j} - \binom{n_1}{j} - \binom{n_2}{j} \right] x^j.$$

Proof. The sum $I_2(G_1;x) + I_2(G_2;x)$ accounts for all co-2-plexes entirely contained in either G_1 or G_2. However, this sum fails to count any co-2-plex S which intersects both G_1 and G_2. Observe that $|S| \leq 2$ for any such co-2-plex. For if not, then without loss of generality, choose $v, w \in S \cap G_1$ and $z \in S \cap G_2$. It follows that $v, w \in N_G(z)$ from the definition of graph join. Therefore, $|N_G[z] \cap S| > 2$, which contradicts that S is a co-2-plex.

Now observe that every set of two or less vertices defines a co-2-plex. G contains $\sum_{j=0}^{2} \binom{n_1+n_2}{j}$ such sets, $\sum_{j=0}^{2} [\binom{n_1}{j} + \binom{n_2}{j}]$ of which are entirely contained in either G_1 or G_2. It follows that $I_2(G;x) = I_2(G_1;x) + I_2(G_2;x) + \sum_{j=0}^{2} \left[\binom{n_1+n_2}{j} - \binom{n_1}{j} - \binom{n_2}{j} \right] x^j$. The final term in the formula for $I_2(G;x)$ adjusts for double counting the empty set as a co-2-plex. □

Given graphs G_1, G_2 with vertices $v_i \in G_i$, $i = 1, 2$, define the *edge join* graph $G = (G_1, v_1) \ominus (G_2, v_2)$ by adding an edge joining v_1 and v_2.

THEOREM 2.3. *If $G = (G_1, v_1) \ominus (G_2, v_2)$, then $I_2(G;x)$ satisfies the following recursive formula*

$$\begin{gathered} I_2(G;x) = x^2 \cdot I_2(G_1 - N[v_1];x) \cdot I_2(G_2 - N[v_2];x) + I_2(G_1;x) \cdot I_2(G_2 - v_2;x) + \\ I_2(G_2;x) \cdot I_2(G_1 - v_1;x) - I_2(G_1 - v_1;x) \cdot I_2(G_2 - v_2;x). \end{gathered}$$

Proof. Let S be a co-2-plex in G, and suppose $v_1, v_2 \in S$. Since v_1v_2 is an edge in G, $N_{G_i}(v_i) \cap S = \varnothing$ for $i = 1, 2$. Therefore, this class of co-2-plexes contributes

$$x^2 \cdot I_2(G - \{N[v_1] \cup N[v_2]\};x)$$

to the total. Notice that $G - \{N[v_1] \cup N[v_2]\} = \{G_1 - N[v_1]\} \cup \{G_2 - N[v_2]\}$ so that Lemma 2.1 implies $x^2 \cdot I_2(G - \{N[v_1] \cup N[v_2]\}; x) = x^2 \cdot I_2(G_1 - N[v_1]; x) \cdot I_2(G_2 - N[v_2]; x)$.

The class where $v_2 \notin S$ contributes $I_2(G - v_2; x)$ to the total, and Lemma 2.1 implies that $I_2(G - v_2; x) = I_2(G_1; x) \cdot I_2(G_2 - v_2; x)$. Similarly, the class where $v_1 \notin S$ contributes $I_2(G_2; x) \cdot I_2(G_1 - v_1; x)$ to the total. The last two classes both include the case where $v_1, v_2 \notin S$, so adjust by subtracting $I_2(G_1 - v_1; x) \cdot I_2(G_2 - v_2; x)$ from the total. □

In Section 3, recursive relationships are used to compute the co-2-plex polynomials of certain families of graphs. The following result is an example of one such relationship.

THEOREM 2.4. *If $K \subseteq G$ is complete, then $I_2(G; x)$ satisfies the following recursion:*

$$I_2(G; x) = \sum_{i=0}^{2} \sum_{S \subseteq K, |S|=i} x^i \cdot I_2(G - \{K \cup N[S]\}; x) + \sum_{v \in K, w \in N(v) \setminus K} x^2 \cdot I_2(G - \{K \cup N[v] \cup N[w]\}; x).$$

Proof. The co-2-plexes in G come in three classes. The first class consists of those co-2-plexes S such that $S \cap K = \varnothing$. This class contributes

$$I_2(G - K; x)$$

to the total. The second class satisfies $|S \cap K| = 2$. In this case, there exists a pair $u, v \in S \cap K$. Since $uv \in E(G)$, deduce that $N(u) \cap S = \{v\}$ and $N(v) \cap S = \{u\}$. It follows that this class contributes

$$x^2 \cdot \sum_{u,v \in K} I_2(G - \{N[u] \cup N[v]\}; x)$$

to the total.

Since $|S \cap K| \leq 2$, it remains to consider those co-2-plexes satisfying $|S \cap K| = 1$. Let $\{v\} = S \cap K$. Notice that either $S \cap N(v) = \varnothing$ or $S \cap N(v) = \{w\}$ for some $w \in V(G) \setminus K$. There are

$$x \cdot \sum_{v \in K} I_2(G - N[v]; x)$$

of the former and

$$x^2 \cdot \sum_{v \in K, w \in N(v) \setminus K} I_2(G - \{N[v] \cup N[w]\}; x)$$

of the latter. The given formula follows by collecting and rearranging terms. □

COROLLARY 2.5. *Given $v \in V(G)$, $I_2(G; x)$ satisfies the following recursion*

$$I_2(G; x) = I_2(G - v; x) + x \cdot I_2(G - N[v]; x) + x^2 \cdot \sum_{w \in N(v)} I_2(G - \{N[v] \cup N[w]\}; x).$$

Proof. Let $K = \{v\}$ and apply the previous result. □

Researchers [15, 16] have studied the first derivative of graph polynomials, e.g. the matching polynomial, independence polynomial, and characteristic polynomial. For example, it is well-known that $\frac{d}{dx}I_1(G;x) = \sum_{v\in V(G)} I_1(G-N[v];x)$. The following theorem is a result on the first derivative of a co-k-plex polynomial.

THEOREM 2.6. *Given integers $k, n \geq 1$,*

$$\frac{d}{dx}I_{k+1}(K_n;x) = n \cdot I_k(K_{n-1};x).$$

Proof. Observe that the complete graph K_n satisfies $I_{k+1}(K_n;x) = \sum_{j=0}^{k+1}\binom{n}{j}x^j$. Therefore,

$$\frac{d}{dx}I_{k+1}(K_n;x) = \sum_{j=0}^{k+1} j\cdot\binom{n}{j}x^{j-1} = n\cdot\sum_{j=0}^{k}\binom{n-1}{j}x^j = n\cdot I_k(K_{n-1};x).$$

□

2.1. Real Roots of the Co-2-plex Polynomial.

In this section, we characterize the graphs such that their co-2-plex polynomial will have all real roots. This class of graphs is simply the class of co-2-plexes. In the proof, we utilize Corollary 2.5 as well as a result of Chudnovsky and Seymour [5]. Before we can offer the result, we need a few more definitions.

Let $f_1(x),\ldots,f_k(x)$ be polynomials in one variable with real coefficients. A set of polynomials is called *compatible* if for all $c_1,\ldots,c_k \geq 0$, all the roots of the polynomial $\sum_{i=1}^{k} c_i f_i(x)$ are real. In addition, the set of polynomials is called *pairwise compatible* if for any two polynomials from the set are compatible. Now, we can state the result of Chudnovsky and Seymour [5].

THEOREM 2.7. *Let $f_1(x),\ldots,f_k(x)$ be polynomials with positive leading coefficients and all roots real. Then $f_1(x),\ldots,f_k(x)$ are pairwise compatible if and only if they are compatible.*

Now, we can prove our main result.

THEOREM 2.8. *Let G be a graph. The co-2-plex polynomial of G has all real roots if and only if G is a co-2-plex.*

Proof. If G is a co-2-plex then $I_2(G;x) = (1+x)^{|V(G)|}$ which has all real roots. Hence, we will concentrate on the other direction.

For the forward direction, we will consider the contrapositive. We will also prove that $I_2(G;x)$ can be written as non-negative combination of co-2-plex polynomials (multiplied by powers of x) of co-2-plex subgraphs of G and that these polynomials are not compatible. Let G be a graph that is not a co-2-plex and let define $C \subseteq V(G)$ as the set of vertices such that $deg(v) > 1 \ \forall v \in C$.

First, consider the case when $C = \{v\}$. Hence, any node induced subgraph of G not containing v is a co-2-plex. Consider the recursive relation of Corollary 2.5 using v. Also, by Theorem 2.7, we just need to show that a pair of these polynomials is not compatible. Hence, consider the compatibility of $I_2(G-v;x)$ and $x^2\cdot I_2(G-\{N[v]\cup N[w]\};x)$ where $w \in N(v)$. Since $C = \{v\}$, then $N[v]\cup N[w] = N[v]$. Thus, the polynomial $I_2(G-v;x) + x^2\cdot I_2(G-\{N[v]\cup N[w]\};x)$ will have -1 as a root, however, other roots will have to satisfy $(1+x)^{deg(v)} = x^2$ which will have imaginary roots. Hence, by Theorem 2.7, the set of polynomials from the recursive formula (each having all real roots) is not compatible.

Now we may assume that the result is true for graphs with $\mid C \mid = t$. Now, assume that $\mid C \mid = t+1$ and let $v \in C$. Once again, $I_2(G;x)$ can be written by the recursion formula using v. By induction, each term of the recursion formula can be written as a non-negative combination of co-2-plex polynomials (multiplied by powers of X) of co-2-plex subgraphs of G and these sets of polynomials are not pairwise compatible. Thus, the entire set of polynomials is not compatible and $I_2(G;x)$ does not have all real roots. □

This result is similar to the result of Chudnovsky and Seymour [5] who showed that the independence polynomial of a graph has all real roots if and only if the graph is claw-free. Indeed, co-2-plexes are 2-claw-free (defined in the next section), however, 2-plexes and paths are also 2-claw-free but their corresponding co-2-plex polynomials have imaginary roots.

3. Examples. This section computes the co-k-plex polynomials for various structured graphs. Most of the results deal with co-2-plex polynomials. First notice that an edgeless graph G on n vertices satisfies $I_k(G;x) = (1+x)^n$ for all $k \geq 1$. Given an integer $k \geq 1$, the graph H is a k-claw if there exists a vertex $u \in V(H)$ such that $N_H[u] = V(H)$, $N(u)$ is a co-k-plex, and $|N(u)| \geq max\{3, k\}$.

Example 1. *If H is a k-claw on n vertices, then*

$$I_k(H;x) = (1+x)^{n-1} + \sum_{i=0}^{k-1} \binom{n-1}{i} x^{i+1}.$$

Proof. The term $(1+x)^{n-1}$ counts all co-k-plexes which exclude the center vertex u. The term $\sum_{i=0}^{k-1}\binom{n-1}{i}x^{i+1}$ counts all those co-k-plexes which include u. □

An r-partite graph can be partitioned into r independent sets. The complete r-partite graph $K_{n_1,\ldots,n_r}$ has all possible edges between distinct partition classes, where $n_1, \ldots, n_r$ are the cardinalities of the partition classes.

Example 2.

$$I_2(K_{n_1,\ldots,n_r};x) = \sum_{i=1}^{r}(1+x)^{n_i} + \sum_{i=1}^{r-1}\sum_{j=0}^{2}\left[\binom{\sum_{p=1}^{i} n_p + n_{i+1}}{j} - \binom{\sum_{p=1}^{i} n_p}{j} - \binom{n_{i+1}}{j}\right] x^j.$$

Proof. The proof is by induction on the number of partition classes r. When $r = 1$, the formula reduces to the correct value of $(1+x)^{n_1}$. Now let $r > 1$ and assume that the formula holds for all $(r-1)$-partite graphs. The induction hypothesis implies that

$$I_2(K_{n_1,\ldots,n_{r-1}};x) = \sum_{i=1}^{r-1}(1+x)^{n_i} + \sum_{i=1}^{r-2}\sum_{j=0}^{2}\left[\binom{\sum_{p=1}^{i} n_p + n_{i+1}}{j} - \binom{\sum_{p=1}^{i} n_p}{j} - \binom{n_{i+1}}{j}\right] x^j.$$

Notice that $K_{n_1,\ldots,n_r}$ can be constructed by performing a graph join between $K_{n_1,\ldots,n_{r-1}}$ and an independent set of cardinality n_r. Theorem 2.2 implies that

$$I_2(K_{n_1,\ldots,n_r};x) = \sum_{i=1}^{r}(1+x)^{n_i} + \sum_{i=1}^{r-2}\sum_{j=0}^{2}\left[\binom{\sum_{p=1}^{i} n_p + n_{i+1}}{j} - \binom{\sum_{p=1}^{i} n_p}{j} - \binom{n_{i+1}}{j}\right] x^j +$$

$$\sum_{j=0}^{2}\left[\binom{\sum_{p=1}^{r-1} n_p + n_r}{j} - \binom{\sum_{p=1}^{r-1} n_p}{j} - \binom{n_r}{j}\right] x^j.$$

The desired formula follows upon simplifying. □

Notice that if $n_i = n$ for all i, then

$$I_2(K_{n,\ldots,n};x) = r(1+x)^n + \sum_{i=1}^{r-1}\sum_{j=0}^{2}\left[\binom{in+n}{j} - \binom{in}{j} - \binom{n}{j}\right]x^j.$$

The path P^n has vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_iv_{i+1} \mid 1 \leq i \leq n-1\}$. By convention, $I_2(P^n;x) = 1$ for all $n \leq 0$. Moreover, it is easy to see that $I_2(P^1;x) = 1+x$ and $I_2(P^2;x) = (1+x)^2$.

Example 3. *For $n \geq 3$, $I_2(P^n;x)$ satisfies the following recursion*

$$I_2(P^n;x) = \sum_{i=1}^{3} x^{i-1} I_2(P^{n-i};x).$$

Proof. Notice that $P^n - v_n = P^{n-1}$, $P^n - N[v_n] = P^{n-2}$, and $P^n - \{N[v_n] \cup N[v_{n-1}]\} = P^{n-3}$. Now Corollary 1 implies $I_2(P^n;x) = I_2(P^{n-1};x) + x \cdot I_2(P^{n-2};x) + x^2 \cdot I_2(P^{n-3};x)$. □

The chordless cycle C^n, where $n \geq 3$, has vertex set $\{v_1, ..., v_n\}$ and edge set $\{v_1v_n\} \cup \{ v_iv_{i+1} \mid 1 \leq i \leq n-1\}$.

Example 4. *For $n \geq 3$, $I_2(C^n;x)$ satisfies the following recursion*

$$I_2(C^n;x) = I_2(P^{n-1};x) + xI_2(P^{n-3};x) + 2x^2 I_2(P^{n-4};x).$$

Proof. Notice that $C^n - v_n = P^{n-1}$, $C^n - N[v_n] = P^{n-3}$, and $C^n - \{N[v_n] \cup N[v_{n-1}]\} = C^n - \{N[v_n] \cup N[v_1]\} = P^{n-4}$. Applying Corollary 1 using v_n gives the following

$$I_2(C^n;x) = I_2(P^{n-1};x) + x \cdot I_2(P^{n-3};x) + x^2 \cdot I_2(P^{n-4};x) + x^2 \cdot I_2(P^{n-4};x).$$

□

A connected and acyclic graph defines a tree. A spider, S_v, is a tree with exactly one vertex v of degree greater than or equal to three.

Example 5. *Let S_v be a spider such that v has degree d. The graph $S - v$ consists of disjoint paths $P^{n_1}, ..., P^{n_r}$ and $I_2(S_v;x)$ satisfies the following recursion*

$$I_2(S_v;x) = \prod_{i=1}^{r} I_2(P^{n_i};x) + x \cdot \left[1 + x \cdot \sum_{j=1}^{d} \frac{I_2(P^{n_j-2};x)}{I_2(P^{n_j-1};x)}\right] \cdot \prod_{i=1}^{r} I_2(P^{n_i-1};x).$$

Proof. The first part of the claim follows from the fact that $\Delta[S-v] \leq 2$. To obtain the recursive formula, apply Corollary 1. By Lemma 2.1,

$$I_2(S_v - v;x) = \prod_{i=1}^{r} I_2(P^{n_i};x).$$

Lemma 2.1 also implies that

$$I_2(S_v - N[v];x) = \prod_{i=1}^{r} I_2(P^{n_i-1};x).$$

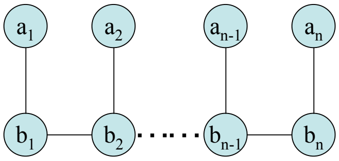

FIG. 3.1. *The centipede W_n.*

It remains to calculate $\sum_{w\in N(v)} I_2(S_v - \{N[v]\cup N[w]\})$. Each neighbor of v belongs to exactly one of the paths $P^{n_1}, ..., P^{n_r}$. Therefore, Lemma 2.1 implies that

$$\sum_{w\in N(v)} I_2(S_v - \{N[v]\cup N[w]\}) = \sum_{j=1}^{d}\left[\frac{I_2(P^{n_j-2};x)}{I_2(P^{n_j-1};x)}\prod_{i=1}^{r} I_2(P^{n_i-1};x)\right].$$

The desired formula follows from Corollary 1. □

A centipede, W_n, is a tree with vertex set $A\cup B = \{a_1, ..., a_n\}\cup\{b_1, ..., b_n\}$ and edge set $\{a_ib_i : 1\le i\le n\}\cup\{b_ib_{i+1} : 1\le i\le n-1\}$. See Figure 3.1. By convention, $I_2(W_n;x) = 1$ for all $n\le 0$. Moreover, it is easy to see that $I_2(W_1;x) = (1+x)^2$ and $I_2(W_2;x) = 1+4x+6x^2+2x^3$.

EXAMPLE 6. *For $n\ge 3$, $I_2(W_n;x)$ satisfies the following recursion*

$$I_2(W_n;x) = (1+x)[x^2\cdot I_2(W_{n-3};x) + x(1+x)\cdot I_2(W_{n-2};x) + I_2(W_{n-1};x)].$$

Proof. Consider the centipede shown in Figure 3.1. Let $W_n[a_n,b_n]$ denote the subgraph induced by $\{a_n,b_n\}$. Notice that $W_n = (W_n[a_n,b_n],b_n)\ominus(W_{n-1},b_{n-1})$. Therefore, applying Theorem 2.3, first compute $I_2(W_n[a_n,b_n]-N_{W_n[a_n,b_n]}[b_n];x)\cdot I_2(W_{n-1}-N_{W_{n-1}}[b_{n-1}];x)$. Observe that $I_2(W_n[a_n,b_n]-N_{W_n[a_n,b_n]}[b_n];x) = I_2(\varnothing;x) = 1$. Now since $W_{n-1}-N_{W_{n-1}}[b_{n-1}] = W_{n-2}[a_{n-2}]\cup W_{n-3}$, Lemma 2.1 implies

$$I_2(W_{n-1}-N_{W_{n-1}}[b_{n-1}];x) = I_2(W_{n-2}[a_{n-2}];x)\cdot I_2(W_{n-3};x) = (1+x)\cdot I_2(W_{n-3};x).$$

Next compute $I_2(W_n[a_n, b_n]-b_n; x)$ and $I_2(W_{n-1}-b_{n-1}; x)$. Clearly, $I_2(W_n[a_n, b_n]-b_n; x) = I_2(W_n[a_n]; x) = (1 + x)$. Since $W_{n-1} - b_{n-1} = W_{n-1}[a_{n-1}] \cup W_{n-2}$, apply Lemma 2.1 to obtain $I_2(W_{n-1} - b_{n-1}; x) = (1 + x) \cdot I_2(W_{n-2}; x)$. In addition, $I_2(W_n[a_n, b_n]; x) = (1 + x)^2$, so the formula from Theorem 2.3 gives

$$I_2(W_n; x) = x^2 \cdot (1 + x) \cdot I_2(W_{n-3}; x) + (1 + x)^2 \cdot (1 + x) \cdot I_2(W_{n-2}; x)+ \\ I_2(W_{n-1}; x) \cdot (1 + x) - (1 + x)^2 \cdot I_2(W_{n-2}; x).$$

The desired formula follows upon simplifying. □

4. Conclusions and Future Work. To conclude, we introduced a polynomial that generalizes the independence polynomial, namely the co-k-plex polynomial. Specifically, we offered examples of the polynomial for a number of graphs. In addition, we proved that the co-2-plex polynomial of a graph has all real roots if and only if the graph is a co-2-plex. This particular result is similar to a result of Seymour and Chudnovsky [5] for the independence polynomial of claw-free graphs. Future work in this area would include generalizing the result for all co-k-plex polynomial for $k > 1$.

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