Lab 3: Subspaces and Manifolds

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1 Introduction

We continue our foray into nonlinear ODE's in this lab. Building off what we know about the various subspaces associated with linear ODE's, we explore the analagous manifolds for nonlinear systems. Where we had the stable subspace E^s, we now have the stable manifold W^s; where we had the unstable subspace E^u, the unstable manifold W^u. How are the linear subspaces and the nonlinear manifolds related? The Stable Manifold Theorem shows that the manifolds at the fixed points of the nonlinear system are tangent to the corresponding subspaces of the linearized ODE. We will show examples of the theorem in action.

2 The Bistable Equation

Consider the nonlinear ODE

$$\begin{bmatrix} u_y \\ v_y \end{bmatrix} = \begin{bmatrix} v \\ cv - f(u) \end{bmatrix}, \tag{1}$$

where

$$f(u) = u(u-a)(1-u), \qquad 0 < a < \frac{1}{2}.$$

The fixed points of this system are, of course, $(0,0)$, $(0,a)$, and $(0,1)$. We construct the linearized versions of (1) using Taylor's Theorem around a fixed point (u_0, v_0):

$$\begin{bmatrix} u \\ v \end{bmatrix} \simeq \begin{bmatrix} 0 & 1 \\ \frac{df}{du}(u_0) & c \end{bmatrix} \begin{bmatrix} u - u_0 \\ v - v_0 \end{bmatrix},$$

where

$$\frac{df}{du}(u_0) = -3u_0^2 + 2(1+a)u_0 - a.$$

We have the following for the $(2,1)$ entry of the matrix in the linearized version:

$$\begin{aligned} u_0 = 0: &\qquad a \\ u_0 = a: &\qquad a^2 - a \\ u_0 = 1: &\qquad 1 - a \end{aligned}$$

For this part of the lab, we fix a and try to find c such that there is a *heteroclinic* connection between the lower and upper fixed points is a curve that lies on the stable manifold of one of the points and the unstable manifold of the other, and vice versa. There is only one such possible heteroclinic connection because flows do not intersect, so if there is a flow joining two fixed points, it is unique. By using PPlane5, we are able to use the bisection method, by hand, to narrow down on the best value of c to achieve a heteroclinic connection. We just input a value of c, plot the stable and unstable orbits of the fixed points at $(0, 0)$ and $(0, 1)$, and try to make them intersect.

In (Table 1) we found that for $a = 0.25$, $c = 0.35355$ produces a heteroclinic connection between the lower and upper critical points. On the left plot, we see the heteroclinic connection along with the eigenvectors of the linearized system. We see that the eigenvectors look to be tangent in a small neighborhood of the fixed points but quickly lose the tangency away from those neighborhoods. In the right plot, we display several trajectories, and we can see how the eigenvectors give good indication of the qualitative features of the phase portrait.

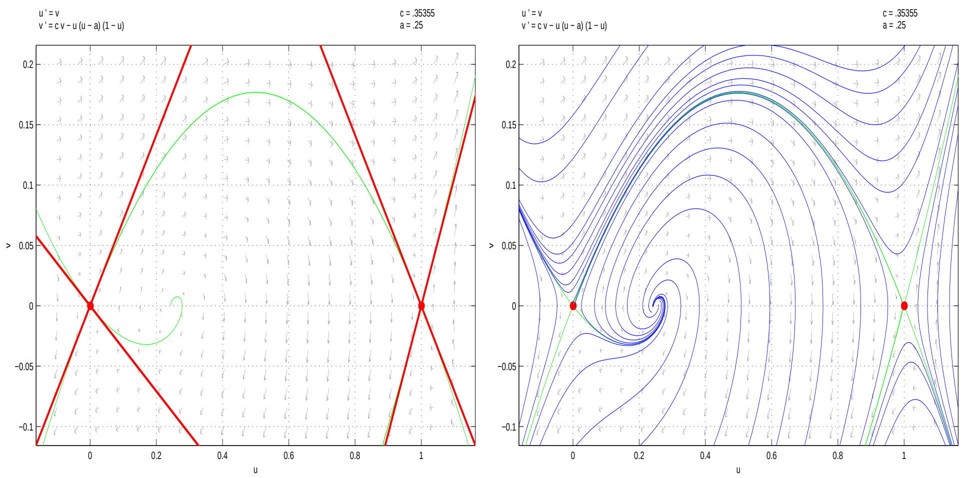

Table 1: (Left) The eigenvectors of the linearized version plotted with the stable and unstable manifolds (Right) The manifolds with several trajectories

3 A Three-Dimensional System

Consider the three-dimensional nonlinear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} x + y^2 \\ x + 2y + 3z^2 \\ x + y^2 - z \end{bmatrix}. \tag{2}$$

The Jacobian of the equations defining (2) is

$$\begin{bmatrix} 1 & 2y & 0 \\ 1 & 2 & 6z \\ 1 & 2y & -1 \end{bmatrix}.$$

Using the `solve` command in Matlab, we find that the fixed points of (2) are $(0,0,0)$ and $(-4,2,0)$. Thus the Jacobians at the fixed points are

$$(0,0,0): \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad (-4,2,0): \begin{bmatrix} 1 & 4 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix}.$$

The eigenvalues and eigenvectors of the Jacobians determine the stable and unstable subspaces of the linearized system, which are locally tangent to the stable and unstable manifolds of the nonlinear system at the fixed points. They are as follow:

$(0,0,0)$:

Stable subspace with eigenvalue -1 and eigenvector $(0,0,1)^T$.

Unstable subspace with eigenvalues 2 and 1 and eigenvectors $(0,1,0)^T$ and $(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})^T$.

$(-4,2,0)$:

Stable subspace with eigenvalues -1 and -0.5616 and eigenvectors $(0,0,1)^T$ and $(0.5984, -0.2336, -0.7664)^T$.

Unstable subspace with eigenvalue 3.5616 and eigenvector $(0.7036, 0.4506, 0.5494)^T$.

4 Conclusion

We have seen how the linearization of a nonlinear system gives a good indication of the stable manifolds near the fixed points of the system: The eigenvectors define a subspace tangent to the manifolds. We explored this concept with a two-dimensional case followed by a three-dimensional one. In addition, we adjusted a parameter value to create a heteroclinic connection among the fixed points of the two-dimensional system.

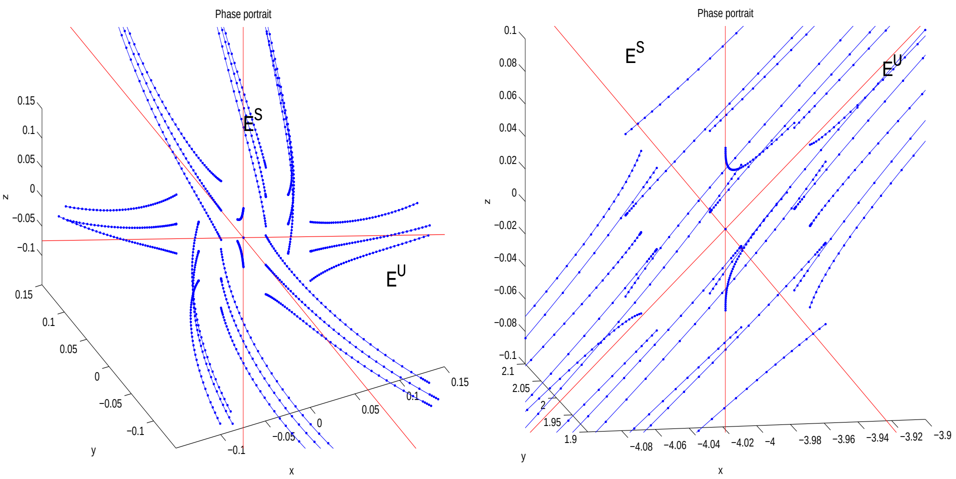

Table 2: The eigenvectors of the linearized version plotted with several trajectories at (Left) $(0, 0, 0)$ (Right) $(-4, 2, 0)$

5 Remarks and Attachments

We used PPlane5 to construct the solutions to the first part of the lab, and we modified pp3d.m to complete the part for the three-dimensional system. In the first part, we held the PPlane5 plots and plotted the eigenvectors over them, using the code attached.