CAAM 336 Practice Final Exam Solutions

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Problem 1.

(a) The eigenvalues of the operator

$$Lu = -\Delta u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

with homogeneous boundary conditions on the unit square can be obtained by substituting the eigenfunctions

$$\phi_{j,k}(x,y) = 2\sin(j\pi x)\sin(k\pi y)$$

into the formula that defines the operator

$$\begin{aligned} L\phi_{j,k} &= -\frac{\partial^2 \phi_{j,k}}{\partial x^2} - \frac{\partial^2 \phi_{j,k}}{\partial y^2} \\ &= -(-2j^2\pi^2 \sin(j\pi x)\sin(k\pi y)) - (-2k^2\pi^2 \sin(j\pi x)\sin(k\pi y)) \\ &= (j^2\pi^2 + k^2\pi^2)\phi_{j,k}(x,y) \end{aligned}$$

So we have

$$L\phi_{j,k} = (j^2\pi^2 + k^2\pi^2)\phi_{j,k}$$

which implies that $\phi_{j,k}$ is an eigenfunction of L with the corresponding eigenvalue

$$\lambda_{j,k} = j^2\pi^2 + k^2\pi^2.$$

(b) We search for the solution of the heat equation

$$u_t = \Delta u$$

in the form

$$u(x,y,t) = \sum_{j,k=1}^{\infty} a_{j,k}(t)\phi_{j,k}(x,y)$$

By substituting this formula into the equation and interchanging the infinite sum with the differentiation operation we get

$$\sum_{j,k=1}^{\infty} \frac{da_{j,k}(t)}{dt}\phi_{j,k}(x,y) = \sum_{j,k=1}^{\infty} a_{j,k}(t)\Delta\phi_{j,k}(x,y)$$

Now we use the fact that $\phi_{j,k}(x,y)$ are the eigenfunctions of $L = -\Delta$ with the eigenvalues $\lambda_{j,k} = j^2\pi^2 + k^2\pi^2$

$$\sum_{j,k=1}^{\infty} \frac{da_{j,k}(t)}{dt}\phi_{j,k}(x,y) = -\sum_{j,k=1}^{\infty} a_{j,k}(t)\lambda_{j,k}\phi_{j,k}(x,y)$$

$$\sum_{j,k=1}^{\infty} \frac{da_{j,k}(t)}{dt}\sin(j\pi x)\sin(k\pi y) = -\sum_{j,k=1}^{\infty} a_{j,k}(t)(j^2\pi^2 + k^2\pi^2)\sin(j\pi x)\sin(k\pi y)$$

We take inner product of both sides with

$$\phi_{m,n}(x,y)$$

and interchanging the inner product with summation we obtain

$$\sum_{j,k=1}^{\infty} \frac{da_{j,k}(t)}{dt}(\phi_{j,k},\phi_{m,n}) = -\sum_{j,k=1}^{\infty} a_{j,k}(t)\lambda_{j,k}(\phi_{j,k},\phi_{m,n})$$

Now we use the fact that $\{\phi_{j,k}(x,y)\}_{j,k=1}^{\infty}$ is an orthonormal system of functions, thus

$$(\phi_{j,k},\phi_{m,n}) = \begin{cases} 0, & \text{if } \quad j \neq m,\ k \neq n \\ 1, & \text{otherwise} \end{cases}$$

Only one term is left on each side

$$\frac{da_{j,k}(t)}{dt} = -\lambda_{j,k}a_{j,k}$$

which is the ODE for $a_{j,k}(t)$.
We can derive the initial conditions for $a_{j,k}(t)$ by representing the initial condition

$$u(x,y,0) = \psi(x,y)$$

as a Fourier series

$$\psi(x,y) = \sum_{j,k=1}^{\infty} b_{j,k}\phi_{j,k}(x,y),$$

where $b_{j,k} = \dfrac{(\psi,\phi_{j,k})}{(\phi_{j,k},\phi_{j,k})} = (\psi,\phi_{j,k})$.
Using the same argument as for the derivation of ODE we get

$$\sum_{j,k}^{\infty} a_{j,k}(0)\phi_{j,k}(x,y) = \sum_{j,k=1}^{\infty} b_{j,k}\phi_{j,k}(x,y)$$

$$a_{j,k}(0) = b_{j,k}$$

which gives us the initial conditions for the ODE.

(c) The solution to the ODE in (b) can be obtained by separation of variables

$$a_{j,k}(t) = Ce^{-\lambda_{j,k}t},$$

where the constant C is determined from the initial condition $a_{j,k}(0) = b_{j,k}$

$$a_{j,k} = b_{j,k}e^{-\lambda_{j,k}t}.$$

Thus the formula for the solution of the heat equation $u(x,y,t)$ becomes

$$u(x,y,t) = \sum_{j,k=1}^{\infty} (\psi, \phi_{j,k}) e^{-\lambda_{j,k} t} \sin(j\pi x) \sin(k\pi y)$$

(d) Since the sine function is bounded by 1 we may write a bound on u

$$|u(x,u,t)| \leq \sum_{j,k=1}^{\infty} |(\psi, \phi_{j,k})| e^{-\lambda_{j,k} t}$$

Remember that $\lambda_{j,k} = j^2\pi^2 + k^2\pi^2 > 0$ increases with $j,k \to \infty$, thus

$$\forall\, j,k \geq 0 \quad e^{-\lambda_{j,k} t} \leq e^{-\lambda_{1,1} t} = e^{-2\pi^2 t}$$

Finally we get

$$|u(x,u,t)| \leq e^{-2\pi^2 t} \sum_{j,k=1}^{\infty} |(\psi, \phi_{j,k})|,$$

implying that $u(x,u,t)$ decays exponentially as $t \to \infty$ at the same rate as $e^{-2\pi^2 t}$.

(e) Here we solve

$$u_t = \Delta u$$

with the boundary conditions

$$\begin{cases} v(x,0) = x \\ v(x,1) = x \\ v(0,y) = 0 \\ v(1,y) = 1 \end{cases}$$

given the function $v(x,y) = x$ that satisfies these boundary conditions.
Suppose that we know the solution u to our problem, then $w = v - u$ must satisfy homogeneous boundary conditions. We also have

$$w_t - \Delta w = (v-u)_t - \Delta(v-u).$$

Since v is linear and does not depend on t we have $\Delta v = 0$ and $v_t = 0$. Using the fact that u solves the heat equation we write

$$w_t - \Delta w = (v_t - \Delta v) - (u_t - \Delta u) = 0 + 0 = 0.$$

So for w we have a homogeneous heat equation with the homogeneous boundary conditions. The initial conditions for w can be obtained from

$$w(x,y,0) = v(x,y) - u(x,y,0) = x - \psi(x,y).$$

Solving the homogeneous heat equation for w with the homogeneous boundary conditions using the formula from (c) and initial conditions $w(x,y,0) = x - \psi(x,y)$ we have the solution of the IBVP with inhomogeneous boundary conditions in the form $u(x,y,t) = v(x,y) - w(x,y,t) = x - w(x,y,t)$.

Problem 2.

(a) We search for the eigenvalues λ_k of the operator

$$Lu = -\frac{d^2u}{dx^2}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(1) = 0$$

in the form

$$\phi_k(x) = A_k \sin\sqrt{\lambda_k}x + B_k \cos\sqrt{\lambda_k}x, \quad k = 1, 2, \ldots$$

Notice that the operator L with the homogeneous Neumann boundary conditions is singular, i.e. there exist a non-zero function ϕ_0 such that $L\phi_0 = -\frac{d^2\phi_0}{dx^2} = 0$. Obviously, the only twice continuously differentiable function that satisfies the equation

$$-\frac{d^2\phi_0}{dx^2} = 0$$

with homogeneous Neumann conditions is a constant function. Since we can normalize the eigenfunctions in any way we like we may take $\phi_0(x) = 1$. So we have found one eigenpair $\lambda_0 = 0,\ \phi_0(x) = 1$.
For $k = 1, 2, \ldots$ we have

$$\frac{d\phi_k}{dx} = A_k\sqrt{\lambda_k}\cos\lambda_k x - B_k\sqrt{\lambda_k}\sin\lambda_k x$$

Using the boundary conditions at $x = 0$ and $x = 1$ we derive a system of two equations for A_k, B_k and λ_k

$$\begin{cases} A_k\sqrt{\lambda_k}\cdot 1 - B_k\sqrt{\lambda_k}\cdot 0 = 0 \\ A_k\sqrt{\lambda_k}\cos\lambda_k - B_k\sqrt{\lambda_k}\sin\lambda_k = 0 \end{cases}$$

this implies that $A_k = 0$ and $\sin\lambda_k = 0$, so for the eigenvalues λ_k we have a relation $\sqrt{\lambda_k} = \pi k$, or $\lambda_k = \pi^2 k^2$. Since we have only 2 equations for 3 unknowns we may choose B_k at our discretion, say $B_k = 1$.
The eigenpairs for $k = 1, 2, \ldots$ are

$$\phi_k(x) = cos(\pi k x), \quad \lambda_k = \pi^2 k^2$$

It is easy to observe that these eigenfunctions form an orthogonal system with respect to the standard L_2 inner product.

(b) We look for a solution to one dimensional wave equation with the homogeneous Neumann boundary conditions in the form

$$u(x,t) = \sum_{k=0}^{\infty} a_k(t)\phi_k(x)$$

We plug this expression in the PDE and interchange the infinite sum operation with differentiation

$$\sum_{k=0}^{\infty} \frac{d^2 a_k}{dt^2}\phi_k = \sum_{k=0}^{\infty} a_k \frac{d^2\phi_k}{dx^2}$$

We now use the fact that $\phi_k(x)$ are the eigenfunctions of $Lu = -\dfrac{d^2u}{dx^2}$, thus

$$\sum_{k=0}^{\infty} \frac{d^2 a_k}{dt^2}\phi_k = -\sum_{k=0}^{\infty} a_k \lambda_k \phi_k$$

Applying the same argument as in 1.(b) we deduce that the ODE for $a_k(t)$ has the form

$$\frac{d^2 a_k(t)}{dt^2} = -\lambda_k a_k(t), \quad k = 0, 1, \ldots$$

As in 1.(b) the initial conditions for the ODEs defining $a_k(t)$ can be obtained by expanding the initial condtions $u(x,0) = \psi(x)$, $\dfrac{\partial u}{\partial t}(x,0) = \gamma(x)$ in the basis of the eigenfunctions

$$\psi(x) = \sum_{k=0}^{\infty} b_k \phi_k, \quad \gamma(x) = \sum_{k=0}^{\infty} c_k \phi_k$$

The argument similar to the one in 1.(b) provides us with the initial condition for ODEs

$$a_k(0) = b_k, \quad \frac{da_k}{dt}(0) = c_k, \quad k = 0, 1, \ldots$$

(c) We notice that the ODEs in the previous section look a lot like the equations which define the eigenfunctions of the second derivative operator. The only difference is that we now solve an initial value problem instead of boundary value problem. We search for the solutions in the form

$$a_k(t) = A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t$$

We use initial conditions to determine coefficients A_k and B_k

$$b_k = a_k(0) = A_k$$

$$c_k = \frac{da_k}{dt}(0) = -A_k\sqrt{\lambda_k}\sin 0 + B_k\sqrt{\lambda_k}\cos 0 = B_k\sqrt{\lambda_k}$$

Notice that the above formulas are valid for $k = 1, 2, \ldots$, we consider the special case $k = 0$ separately

$$\frac{d^2 a_0}{dt^2} = -\lambda_0 a_0 = 0$$

so it must be that $a_0(t) = A_0 t + B_0$, where we determine A_0, B_0 from

$$a_0(0) = B_0 = b_0$$

$$\frac{d^2 a_0}{dt^2}(0) = A_0 = c_0$$

(d) Using the formulas from the previous two sections we derive an expression for the solution to one dimensional wave equation with the homogeneous Neumann boundary conditions

$$u(x,t) = b_0 + c_0 t + \sum_{k=1}^{\infty}(b_k \cos\sqrt{\lambda_k}t + \frac{c_k}{\sqrt{\lambda_k}}\sin\sqrt{\lambda_k}t)\cos\pi kx$$

We observe that the solution has a term which grows or decreases (depending on the sign of c_0) linearly as $t \to \infty$ as well as a term which keeps oscillating as the solution evolves in time.

For the heat equation with homogeneous Neumann boundary conditions the solution has the form

$$u(x,t) = b_0 + \sum_{k=1}^{\infty} b_k e^{-\lambda_k t} \cos \pi k x,$$

where b_k has the same meaning as for the wave equation IBVP $b_k = \dfrac{(\psi, \phi_k)}{(\phi_k, \phi_k)} = 2\int\limits_0^1 \psi(x) cos(\pi k x) dx$ for $k = 1, 2, \ldots$ and $b_0 = \dfrac{(\psi, \phi_0)}{(\phi_0, \phi_0)} = \int\limits_0^1 \psi(x) dx$ is an average of the initial profile over the interval $[0,1]$. In contrast to the solution to wave equation, the solution of the heat equation with the same bounudary conditions has a limit as $t \to \infty$, $u(x,t) \to b_0$. The solutions flattens out to an average of the initial data.

Problem 3.

(a) Fredholm alternative tells us that for an equation $Ax = b$ with a singular linear operator A we have exactly two possibilities

- If $b \in \text{Range}(A)$, then $Ax = b$ has infinitely many solutions.
- If $b \notin \text{Range}(A)$, then $Ax = b$ has no solutions.

We see that the case when A is singular and $Ax = b$ has a unique solution is impossible. Indeed, suppose that A is singular, hence $\text{Null}A \neq \emptyset$. Let x_0 satisfy $Ax_0 = b$, then for y in the null space of A we have $A(x_0 + y) = Ax_0 + Ay = b + 0$, so $x_0 + y$ also solves $Ax = b$. Thus, existence of solution of $Ax = b$ implies that infinite number of solutions exist.

In our case A is given by

$$A = \begin{pmatrix} -500 & 500 \\ 500 & -500 \end{pmatrix}$$

$$Ax = \begin{pmatrix} -500x_1 + 500x_2 \\ 500x_1 - 500x_2 \end{pmatrix}$$

One can easily observe that

$$\text{Range}(A) = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix}, \quad \text{Null}(A) = \begin{pmatrix} \beta \\ \beta \end{pmatrix}, \quad \alpha \in \mathbb{R},\ \beta \in \mathbb{R}$$

Thus if $b \in \text{Range}(A)$ $\exists \alpha \in \mathbb{R}$ $b = \begin{pmatrix} -\alpha \\ \alpha \end{pmatrix}$. Then we can find solutions of $Ax = b$ as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfying $x_1 - x_2 = \dfrac{\alpha}{500}$. We can take $x_0 = \begin{pmatrix} \frac{\alpha}{500} \\ 0 \end{pmatrix}$ and generate a general solution by adding any vector from the null space of A.

$$x = \begin{pmatrix} \frac{\alpha}{500} \\ 0 \end{pmatrix} + \begin{pmatrix} \beta \\ \beta \end{pmatrix}$$

(b) We compute matrix exponential e^{tA} using diagonalization of A. The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -1000$. Corresponding eigenvectors are

$$q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Since A is symmetric, we can represent A in factored form $A = QDQ^T$, where $Q = (q_1, q_2)$ is an orthogonal matrix and

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -1000 \end{pmatrix}$$

We use the fact that for diagonalizable A we can represent matrix exponential as $e^{tA} = e^{tQDQ^T} = Qe^{tD}Q^T$. This results in

$$e^{tA} = \frac{1}{2}\begin{pmatrix} 1+e^{-1000t} & 1-e^{-1000t} \\ 1-e^{-1000t} & 1+e^{-1000t} \end{pmatrix}$$

We observe that as $t \to \infty$, $e^{tA} \to I$, where I is 2×2 identity matrix.

(c) We consider an ODE in vector form

$$\frac{d}{dt}y(t) = Ay(t)$$

For forward Euler we use the approximation

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta t} = Ay(t_k).$$

The iteration is

$$y(t_{k+1}) = (I + \Delta t A)y(t_k).$$

For backward Euler we approximate the ODE by

$$\frac{y(t_{k+1}) - y(t_k)}{\Delta t} = Ay(t_{k+1}).$$

At each iteration we have to solve a linear system

$$(I - \Delta A)y(t_{k+1}) = y(t_k),$$

so the expression for $y(t_k)$ becomes

$$y(t_{k+1}) = (I - \Delta A)^{-1}y(t_k).$$

(d) We have proved that $e^{tA} \to I$ as $t \to \infty$. The solution to the ODE with initial conditions $y(0) = y_0$ is given by $y(t) = e^{tA}y_0$, so if we want the numerical solution to mimic the behavior of the true solution we must have $y(t_k)$ to stay bounded for any initial guess y_0.

For forward Euler we express the solution at time step k in terms of initial value y_0 as

$$y(t_k) = (I + \Delta t A)^k y_0$$

We use the fact that $(I + \Delta t A)^k$ stays bounded as $k \to \infty$ if all the eigenvalues of $(I + \Delta t A)$ are inside the unit circle on the complex plane. If we denote by $\lambda(A)$ the eigenvalue of A we should have

$$\lambda(I + \Delta t A) = 1 + \lambda(\Delta t A) = 1 + \Delta \lambda(A)$$

In our case $\lambda(A) = \{0, -1000\}$, so we must have

$$|1 + \Delta \cdot 0| \le 1, \quad |1 - \Delta \cdot 1000| \le 1,$$

so the condition for the time step Δt is

$$\Delta t \leq \frac{2}{1000} = \frac{1}{500}$$

For backward Euler we have

$$y(t_k) = ((I - \Delta t A)^{-1})^k y_0.$$

The relation for the eigenvalues becomes

$$\lambda((I + \Delta t A)^{-1}) = \frac{1}{1 - \Delta t \lambda(A)}$$

Substituting $\lambda(A) = \{0, -1000\}$ we get

$$\frac{1}{|1 - \Delta \cdot 0|} \leq 1, \quad \frac{1}{|1 + \Delta \cdot 1000|} \leq 1.$$

Observing that the inequalities above are valid for any positive Δt we deduce that backward Euler method is unconditionally stable for our problem.

Problem 4.

(a) Consider the PDE

$$u_t(x,t) - u_{xx}(x,t) - \epsilon u(x,t) = f(x,t)$$

with homogeneous Dirichlet boundary conditions, then if we take the inner product of both sides with the test function $v(x)$ that satisfies the same boundary conditions, we integrate the middle term on the left by parts to get the weak form

$$(u_t, v) + (u_x, v_x) - \epsilon(u, v) = (f, v), \quad \forall\, v \in C^2[0,1] : \; v(0) = v(1) = 0$$

(b) Now if we consider some finite functional space V_N with the basis $\{\phi_1, \dots, \phi_N\}$ we can expand our approximate solution u_N in this basis

$$u_N(x,t) = \sum_{k=1}^{N} \alpha_k(t)\phi_k(x)$$

we apply Galerkin method to the weak form

$$(\frac{d}{dt}\sum_{k=1}^{N} \alpha_k(t)\phi_k, \phi_j) + (\frac{d}{dx}\sum_{k=1}^{N} \alpha_k(t)\phi_k, \frac{d}{dx}\phi_j) - \epsilon(\sum_{k=1}^{N} \alpha_k(t)\phi_k, \phi_j) = (f, \phi_j)$$

$$\sum_{k=1}^{N} \frac{d}{dt}\alpha_k(t)(\phi_k, \phi_j) + \sum_{k=1}^{N} \alpha_k(t)(\frac{d}{dx}\phi_k, \frac{d}{dx}\phi_j) - \epsilon \sum_{k=1}^{N} \alpha_k(t)(\phi_k, \phi_j) = (f, \phi_j),$$

where $(.,.)$ is a standard L_2 inner product with respect to $x \in [0,1]$.

(c) I we let $\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_N(t) \end{pmatrix}$, then we can rewrite the equations in the previous section in matrix form as

$$A\frac{d}{dt}\alpha(t) + B\alpha(t) = f(t),$$

where $A_{ij} = (\phi_i, \phi_j)$, $B_{ij} = (\frac{d}{dx}\phi_i, \frac{d}{dx}\phi_j) - \epsilon(\phi_i, \phi_j)$,
$f(t) = \begin{pmatrix} (f(x,t), \phi_1) \\ \vdots \\ (f(x,t), \phi_N) \end{pmatrix}$. We can derive the initial conditions for the system of ODEs above by projecting the initial condition of the continuous problem $u(x, 0) = \psi(x)$ on the finite dimensional space V_N. We take $\psi_N(x) = \sum_{k=1}^{N} \beta_k \phi_k(x)$, where we get the vector of coefficients $\beta = (\beta_1, \ldots, \beta_N)^T$ by solving the projection problem $G\beta = \hat{\psi}$, where $G_{ij} = (\phi_i, \phi_j)$ is Gram matrix, and $\hat{\psi}_k = (\psi, \phi_k)$ is the right-hand side.

(d) We apply backward Euler method to our problem

$$A\frac{\alpha(t_{k+1}) - \alpha(t_k)}{\Delta t} + B\alpha(t_{k+1}) = f(t_{k+1})$$

$$(A + \Delta t B)\alpha(t_{k+1}) = A\alpha(t_k) + \Delta t f(t_{k+1})$$

$$\alpha(t_{k+1}) = (A + \Delta t B)^{-1}(A\alpha(t_k) + \Delta t f(t_{k+1}))$$

(e) The finite element solution does not depend on the choice of the basis functions ϕ_k but only on the space V_N.

(f) If we solve the problem with inhomogeneous Dirichlet boundary conditions $u(0, t) = a$, $u(1, t) = b$ we have to add to the basis two additional functions ϕ_0, ϕ_{N+1} such that, for example $\phi_0(0) = 1$, $\phi_{N+1}(1) = 1$. Our finite element solution then has the form

$$u_N(x, t) = a\phi_0(x) + b\phi_{N+1}(x) + \sum_{k=1}^{N} \alpha_k(t)\phi_k(x)$$

We modify the right-hand side of the finite element equations accordingly

$$f(t) = \begin{pmatrix} (f(x,t), \phi_1) - a(\phi_0, \phi_1) - b(\phi_{N+1}, \phi_1) \\ \vdots \\ (f(x,t), \phi_N) - a(\phi_0, \phi_N) - b(\phi_{N+1}, \phi_N) \end{pmatrix}$$