CAAM 336
DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 1

Posted 20 February 2008, 10:00 AM.

Due 12 noon on Wednesday, 27 February 2008.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are five questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [20 points]
   Vector spaces

   (a) Determine if the following sets are vector spaces or not- justify your answer. (Define addition and scalar multiplication in the obvious way.)

      i. $\left\{ f \in C[0,1] : \int_0^1 f(x)\,dx = 0 \right\}$.

      ii. $\{f \in C[0,1] : f = \frac{d^2u}{dx^2},$ for some $u \in C^2[0,1]\}$.

      iii. $\{f \in C[0,1] : \min_{x\in[0,1]} f(x) = 0\}$.

      iv. $\{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| = 1\}$.

   (b) A real inner product is a function that takes two vectors from a vector space V to produce a real number and satisfies what three properties?

   (c) Write down a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^2$. Assume you are using the canonical $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ basis.

   (d) Determine the nullspace of the following operators

      i. $L : C^2[0,1] \to C[0,1]$ such that

$$Lu = -\frac{d^2u}{dx^2}$$

      ii. $L_D : C_D^2[0,1] \to C[0,1]$ where $C_D^2[0,1] = \left\{ f \in C^2[0,1] : f(0) = f(1) = 0 \right\}$ and

$$L_D u = -\frac{d^2u}{dx^2}$$

      iii. $L_N : C_N^2[0,1] \to C[0,1]$ where $C_N^2[0,1] = \left\{ f \in C^2[0,1] : \frac{df}{dx}(0) = \frac{df}{dx}(1) = 0 \right\}$ and

$$L_N u = -\frac{d^2u}{dx^2}$$

2. [20 points]
   Consider the matrix and vector

$$\mathbf{A} = \begin{pmatrix} -1/2 & -7/2 & 1 \\ -7/2 & -1/2 & 1 \\ 1 & 1 & -5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   (a) Compute the eigenvalues and eigenvectors of $\mathbf{A}$.

   (b) Solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ by using the spectral method.

3. [20 points]
   Approximation of functions

   (a) Consider the function $f(x) = \sqrt{x}$. Find the best approximation to $f(x)$ with the L^2 inner product on the interval $[0, 1]$ from $\mathbb{P}_3$, the space of third-order polynomials, using the following orthonormal basis:

   $$\left\{1, 2\sqrt{3}\left(x - \frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right), \sqrt{7}(20x^3 - 30x^2 + 12x - 1)\right\}$$

   (b) Find the equation of the line which best approximates $g(x) = \cos(x)$ at the points $x = 0, \pi/4, \pi/2, 3\pi/4, \pi$ in the Euclidean norm on $\mathbb{R}^5$.

4. [20 points]
Solve the boundary value problem

$$-\frac{d^2u}{dx^2} = \sin(2\pi x)$$

$$u(0) = -1 \qquad u(2) = 3.$$

by using Fourier series, shifting the data if necessary.

5. [20 points]
Consider the equation,

$$-\frac{d}{dx}\left(\frac{1}{1+x}\frac{du}{dx}(x)\right) = x, \qquad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

(a) Find the analytic solution to the differential equation; that is, the solution you obtain through direct integration.

(b) Write down the weak form of the of the boundary value problem given above.

(c) Show that the energy inner product derived from the weak form of the boundary value problem is in fact an inner product.

(d) Suppose we take for $\phi_1, \ldots, \phi_N$ the standard piecewise linear 'hat' functions on the uniform mesh $h = 1/(N+1)$, $x_k = kh$,

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

Set up (but do not solve) the system $\mathbf{Ku} = \mathbf{f}$. Specifically, calculate $\mathbf{K}$ and $\mathbf{f}$. You can either calculate them in general or for $N = 4$, whichever you like.

(e) Describe how $\mathbf{u}$, the solution of the system $\mathbf{Ku} = \mathbf{f}$, approximates the function u, the solution to the boundary value problem. In particular, how can you compare them to each other?