CAAM 336

DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 1

SOLUTIONS

1. [20 points]
   Vector spaces

   (a) Determine if the following sets are vector spaces or not- justify your answer. (Define addition and scalar multiplication in the obvious way.)

   i. $\left\{ f \in C[0,1] : \int_0^1 f(x)\,dx = 0 \right\}$.
   ii. $\{f \in C[0,1] : f = \frac{d^2u}{dx^2},$ for some $u \in C^2[0,1]\}$.
   iii. $\{f \in C[0,1] : \min_{x\in[0,1]} f(x) = 0\}$.
   iv. $\{x \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| = 1\}$.

   Solution.

   i. This is a vector space. It is the nullspace of the linear operator defined by the definite integral from 0 to 1.
   ii. This is a vector space. It is the range of the linear operator defined by differentiating twice.
   iii. This is not a vector space. The function $f(x) = x$ is in this space, but $-f$ is not, since the minimum of $-f$ is -1.
   iv. This is not a vector space. $(0, 0, 1)$ is in the space, but two times it is not.

   (b) A real inner product is a function that takes two vectors from a vector space V to produce a real number and satisfies what three properties?

   Solution. We will denote the inner product of u and v by $\langle u, v\rangle$. The three properties are:

   i. $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$
   ii. $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.
   iii. $\langle u, u\rangle \geq 0$ for all $u \in V$, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

   (c) Write down a linear operator from $\mathbb{R}^3$ to $\mathbb{R}^2$. Assume you are using the canonical $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ basis.

   Solution. All you need to do is write down any 2×3 matrix.

   (d) Determine the nullspace of the following operators

i. $L : C^2[0,1] \to C[0,1]$ such that

$$Lu = -\frac{d^2u}{dx^2}$$

ii. $L_D : C_D^2[0,1] \to C[0,1]$ where $C_D^2[0,1] = \{f \in C^2[0,1] : f(0) = f(1) = 0\}$ and

$$L_D u = -\frac{d^2u}{dx^2}$$

iii. $L_N : C_N^2[0,1] \to C[0,1]$ where $C_N^2[0,1] = \left\{f \in C^2[0,1] : \frac{df}{dx}(0) = \frac{df}{dx}(1) = 0\right\}$ and

$$L_N u = -\frac{d^2u}{dx^2}$$

Solution.

i. The nullspace is all linear functions, i.e., functions of the form $u(x) = ax + b$ for some $a, b \in \mathbb{R}$
ii. The nullspace is just the zero function.
iii. The nullspace is all constants.

2. [20 points]
Consider the matrix and vector

$$\mathbf{A} = \begin{pmatrix} -1/2 & -7/2 & 1 \\ -7/2 & -1/2 & 1 \\ 1 & 1 & -5 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Compute the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Solve $\mathbf{Ax} = \mathbf{b}$ by using the spectral method.

Solution. In order to use the spectral method, you must first find the eigenvalues and eigenvectors of $\mathbf{A}$. The first step in doing this is to find the characteristic polynomial, that is, $\det(\lambda\mathbf{I} - \mathbf{A})$.

$$
\begin{aligned}
\det(\lambda\mathbf{I}-\mathbf{A}) &= \det\begin{pmatrix}\lambda+1/2 & 7/2 & -1\\ 7/2 & \lambda+1/2 & -1\\ -1 & -1 & \lambda+5\end{pmatrix}\\
&= (\lambda+1/2)\det\begin{pmatrix}\lambda+1/2 & -1\\ -1 & \lambda+5\end{pmatrix} - 7/2\det\begin{pmatrix}7/2 & -1\\ -1 & \lambda+5\end{pmatrix} - 1\det\begin{pmatrix}7/2 & \lambda+1/2\\ -1 & -1\end{pmatrix}\\
&= (\lambda+1/2)\,[(\lambda+1/2)(\lambda+5)-1] - 7/2\,[7/2(\lambda+5)-1] - 1\,[-7/2+\lambda+1/2]\\
&= \left(\lambda+\frac{1}{2}\right)\left(\lambda^2+\frac{11}{2}\lambda+\frac{3}{2}\right) - \frac{7}{2}\left(\frac{7}{2}\lambda+\frac{33}{2}\right) - 1(\lambda-3)\\
&= \lambda^3+\frac{11}{2}\lambda^2+\frac{3}{2}\lambda+\frac{1}{2}\lambda^2+\frac{11}{4}\lambda+\frac{3}{4}-\frac{49}{4}\lambda-\frac{231}{4}-\lambda+3\\
&= \lambda^3+6\lambda^2+\left(\frac{3}{2}+\frac{11}{4}-\frac{49}{4}-1\right)\lambda+\left(\frac{3}{4}-\frac{231}{4}+3\right)\\
&= \lambda^3+6\lambda^2-9\lambda-54\\
&= \lambda^2(\lambda+6)-9(\lambda+6)\\
&= (\lambda-3)(\lambda+3)(\lambda+6)
\end{aligned}
$$

We now know the three eigenvalues, we must find the eigenvectors.

First, examine the matrix for $\lambda_1 = 3$:

$$3\mathbf{I}-\mathbf{A} = \begin{pmatrix}7/2 & 7/2 & -1\\ 7/2 & 7/2 & -1\\ -1 & -1 & 8\end{pmatrix}$$

The eigenvector here is $\mathbf{u}_1 = (1,-1,0)^T$.

Next, $\lambda_2 = -3$:

$$-3\mathbf{I}-\mathbf{A} = \begin{pmatrix}-5/2 & 7/2 & -1\\ 7/2 & -5/2 & -1\\ -1 & -1 & 2\end{pmatrix}$$

The eigenvector here is $\mathbf{u}_2 = (1,1,1)^T$.

Finally, the matrix for $\lambda_3 = -6$:

$$-6\mathbf{I}-\mathbf{A} = \begin{pmatrix}-11/2 & 7/2 & -1\\ 7/2 & -11/2 & -1\\ -1 & -1 & -1\end{pmatrix}$$

The final eigenvector is $\mathbf{u}_3 = (1,1,-2)^T$.

The solution to the equation $\mathbf{Ax} = \mathbf{b}$ is then

$$\begin{aligned}
\mathbf{x} &= \frac{\langle \mathbf{u}_1, \mathbf{b}\rangle}{\lambda_1\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2, \mathbf{b}\rangle}{\lambda_2\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 + \frac{\langle \mathbf{u}_3, \mathbf{b}\rangle}{\lambda_3\langle \mathbf{u}_3, \mathbf{u}_3\rangle}\mathbf{u}_3 \\
&= \frac{1}{(3)(2)}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{2}{(-3)(3)}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-1}{(-6)(6)}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{6} - \frac{2}{9} + \frac{1}{36} \\ -\frac{1}{6} - \frac{2}{9} + \frac{1}{36} \\ -\frac{2}{9} - \frac{1}{18} \end{pmatrix} \\
&= \boxed{\begin{pmatrix} -1/36 \\ -13/36 \\ -5/18 \end{pmatrix}}.
\end{aligned}$$

3. [20 points]
Approximation of functions

(a) Consider the function $f(x) = \sqrt{x}$. Find the best approximation to $f(x)$ with the L^2 inner product on the interval $[0, 1]$ from $\mathbb{P}_3$, the space of third-order polynomials, using the following orthonormal basis:

$$\left\{1, 2\sqrt{3}\left(x - \frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right), \sqrt{7}(20x^3 - 30x^2 + 12x - 1)\right\}$$

Solution. The best approximation to f(x) arrives from a solution to the matrix vector equation arriving from the Gram matrix, based on the basis vectors $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ are as follows:

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= 2\sqrt{3}\,(x - 1/2) \\
\phi_2 &= 6\sqrt{5}\,(x^2 - x + 1/6) \\
\phi_3 &= \sqrt{7}\,(20x^3 - 30x^2 + 12^x - 1)
\end{aligned}$$

Thus our matrix equation to solve is

$$Gc = F$$

where

$$\begin{aligned}
G_{ij} &= (\phi_i, \phi_j) \\
F_j &= (f, \phi_j)
\end{aligned}$$

However since our basis is orthonormal, our Gram matrix is the identity. Thus the best approximation $p_3(x)$ to $f(x)$ from $\mathcal{P}_3$ is

$$p_3(x) = \sum_{i=0}^{3} c_i \phi_i(x)$$

where $c_i = (f, \phi_i)$. That is

$$\begin{aligned}
c_i &= (f, \phi_i) \\
&= \int_0^1 \sqrt{x}\phi_i \, dx
\end{aligned}$$

Thus

$$\begin{aligned} c_0 &= \int_0^1 \sqrt{x}\, dx \\ &= 2/3 \end{aligned}$$

$$\begin{aligned} c_1 &= 2\sqrt{3} \int_0^1 \sqrt{x}\,(x - 1/2)\; dx \\ &= 2\sqrt{3}(2/5 - 1/3) \end{aligned}$$

$$\begin{aligned} c_2 &= 6\sqrt{5} \int_0^1 \sqrt{x}\,\left(x^2 - x + 1/6\right)\; dx \\ &= 6\sqrt{5}(2/7 - 2/5 + 1/9) \end{aligned}$$

$$\begin{aligned} c_3 &= \sqrt{7} \int_0^1 \sqrt{x}\,\left(20x^3 - 30x^2 + 12^x - 1\right)\; dx \\ &= \sqrt{7}(40/9 - 60/7 + 24/5 - 2/3) \end{aligned}$$

(b) Find the equation of the line which best approximates $g(x) = \cos(x)$ at the points $x = 0, \pi/4, \pi/2, 3\pi/4, \pi$ in the Euclidean norm on $\mathbb{R}^5$.

Solution. Here we want to solve the following problem

$$\min_{a,b\in R^3} \left\| \begin{pmatrix} b + ax_1 \\ b + ax_2 \\ b + ax_3 \\ b + ax_4 \\ b + ax_5 \end{pmatrix} - \begin{pmatrix} \cos(x_1) \\ \cos(x_2) \\ \cos(x_3) \\ \cos(x_4) \\ \cos(x_5) \end{pmatrix} \right\|_2$$

which is equivalent to solving the matrix vector equation

$$Gc = F$$

where

$$\begin{aligned} G &= \begin{bmatrix} \sum_{i=1}^5 1 & \sum_{i=1}^5 x_i \\ \sum_{i=1}^5 x_i & \sum_{i=1}^5 x_i^2 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 5\pi/2 \\ 5\pi/2 & 30\pi^2/16 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} F &= \begin{bmatrix} \sum_{i=1}^5 \cos(x_i) \\ \sum_{i=1}^5 x_i \cos(x_i) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\pi/(2\sqrt{2}) - \pi \end{bmatrix} \end{aligned}$$

$$c = \begin{bmatrix} b \\ a \end{bmatrix}$$

This gives us that c is given by

$$\begin{aligned} c &= G^{-1}F \\ &= \frac{1}{5(30\pi^2/16) - 25\pi^2/4}\begin{bmatrix} 5\pi/2(\pi/2\sqrt{2}+\pi) \\ -5((\pi/2\sqrt{2}+\pi) \end{bmatrix} \end{aligned}$$

And we get that the least squares fit line is $l(x) = c(1) + c(2)x$

4. [20 points]
Solve the boundary value problem

$$-\frac{d^2u}{dx^2} = \sin(2\pi x)$$

$$u(0) = -1 \qquad u(2) = 3.$$

by using Fourier series, shifting the data if necessary.

Solution. Since the right hand side function is a single term in the Fourier series, we get that the solution is given by a single fourier term and a line that shifts the solution to fit the boundary conditions. That is we get

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \frac{\langle \sin(2\pi x), \sin(\frac{n\pi x}{2})\rangle}{\langle \sin(\frac{n\pi x}{2}), \sin(\frac{n\pi x}{2})\rangle} \sin\left(\frac{n\pi x}{2}\right) + 2x - 1 \\ &= \frac{\sin(2\pi x)}{4\pi^2} + 2x - 1 \end{aligned}$$

5. [20 points]
Consider the equation,

$$-\frac{d}{dx}\left(\frac{1}{1+x}\frac{du}{dx}(x)\right) = x, \qquad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

(a) Find the analytic solution to the differential equation; that is, the solution you obtain through direct integration.

Solution. Go ahead and multiply both side by -1, then integrate. You get

$$\frac{1}{1+x}\frac{du}{dx}(x) = -\frac{1}{2}x^2 + C_0$$

Multiply both sides by $1+x$

$$\begin{aligned} \frac{du}{dx}(x) &= (1+x)\left(-\frac{1}{2}x^2 + C_0\right) \\ &= -\frac{1}{2}x^2 + C_0 - \frac{1}{2}x^3 + C_0 x \end{aligned}$$

and integrate again.

$$u(x) = -\frac{1}{8}x^4 - \frac{1}{6}x^3 + \frac{C_0}{2}x^2 + C_0 x + C_1$$

Now plug in the boundary conditions. As always, the easy one to do first is $u(0) = 0$. This one immediately tells you that $C_1 = 0$. Now plut in $u(1) = 0$:

$$0 = -\frac{1}{8} - \frac{1}{6} + \frac{C_0}{2} + C_0$$

Solving for C_0, you get that $C_0 = 7/36$. Not terribly pleasant, but you don't have to do anything else with this other than write down the answer, so let's do it.

$$\boxed{u(x) = -\frac{1}{8}x^4 - \frac{1}{6}x^3 + \frac{7}{72}x^2 + \frac{7}{36}x}$$

(b) Write down the weak form of the of the boundary value problem given above.

Solution. The weak form is: find $u \in C_D^2[0,1]$ such that

$$a(u,v) = \langle f, v\rangle \text{ for all } v \in C_D^2[0,1],$$

where $f(x) = x$. Alternatively, you could put it in the integral form: find $u \in C_D^2[0,1]$ such that

$$\int_0^1 \frac{1}{1+x}\frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx = \int_0^1 xv(x)\,dx \text{ for all } v \in C_D^2[0,1]$$

(c) Show that the energy inner product derived from the weak form of the boundary value problem is in fact an inner product.

Solution. The energy inner product is $a(u,v)$, above. Let's go through the rules.

- Rule 1: Linearity in the first slot

$$\begin{aligned} a(\alpha u + \beta v, w) &= \int_0^1 \frac{1}{1+x}\frac{d(\alpha u + \beta v)}{dx}(x)\frac{dw}{dx}(x)\,dx \\ &= \int_0^1 \frac{1}{1+x}\left(\alpha\frac{du}{dx}(x) + \beta\frac{dv}{dx}(x)\right)\frac{dw}{dx}(x)\,dx \\ &= \alpha\int_0^1 \frac{1}{1+x}\frac{du}{dx}(x)\frac{dw}{dx}(x)\,dx + \beta\int_0^1 \frac{1}{1+x}\frac{dv}{dx}(x)\frac{dw}{dx}(x)\,dx \\ &= \alpha a(u,w) + \beta a(v,w) \end{aligned}$$

- Rule 2: Symmetry

$$\begin{aligned} a(u,v) &= \int_0^1 \frac{1}{1+x}\frac{du}{dx}(x)\frac{dv}{dx}(x)\,dx \\ &= \int_0^1 \frac{1}{1+x}\frac{dv}{dx}(x)\frac{du}{dx}(x)\,dx \\ &= a(v,u) \end{aligned}$$

- Rule 3: Positive-definiteness

$$\begin{aligned} a(u,u) &= \int_0^1 \frac{1}{1+x}\left(\frac{du}{dx}(x)\right)^2\,dx \\ &\geq 0 \end{aligned}$$

  because $1/(1+x) > 0$ on the interval $[0,1]$, and something squared is always non-negative.
  Furthermore, if $a(u,u) = 0$, since $1/(1+x) > 0$, this demands that $u' \equiv 0$ on the interval $[0,1]$, which means it's a constant. But since u must satisfy homogeneous Dirichlet conditions, this means that $u \equiv 0$.

(d) Suppose we take for $\phi_1, \dots, \phi_N$ the standard piecewise linear 'hat' functions on the uniform mesh $h = 1/(N+1)$, $x_k = kh$,

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

Set up (but do not solve) the system $\mathbf{Ku} = \mathbf{f}$. Specifically, calculate $\mathbf{K}$ and $\mathbf{f}$. You can either calculate them in general or for $N = 4$, whichever you like.

Solution. First lets find $\mathbf{K}$. We know that this will be a tridiagonal matrix, because of the properties of the hat functions. It's also symmetric, which means we ultimately have two basic inner products to find: the ones on the diagonal, and the ones just one off the diagonal. Let's do the diagonal ones first:

$$\begin{aligned} K_{ii} &= a(\phi_i, \phi_i) \\ &= \int_0^1 \frac{1}{1+x}\left(\frac{d\phi_i}{dx}(x)\right)^2\,dx \\ &= \int_{h(i-1)}^{h(i+1)} \frac{1}{1+x}\left(\frac{1}{h}\right)^2\,dx \\ &= \frac{1}{h^2}\int_{h(i-1)}^{h(i+1)} \frac{1}{1+x}\,dx \\ &= \frac{1}{h^2}\Big[\ln(1+x)\Big]_{h(i-1)}^{h(i+1)} \\ &= \frac{1}{h^2}\Big[\ln(1+h(i+1)) - \ln(1+h(i-1))\Big] \\ &= \frac{1}{h^2}\ln\left(\frac{1+h(i+1)}{1+h(i-1)}\right) \end{aligned}$$

Now we find the off-diagonal elements

$$\begin{aligned} K_{i,i+1} &= a(\phi_i, \phi_{i+1}) \\ &= \int_0^1 \frac{1}{1+x}\frac{d\phi_i}{dx}(x)\frac{d\phi_{i+1}}{dx}(x)\,dx \\ &= \int_{ih}^{h(i+1)} \frac{1}{1+x}\frac{-1}{h^2}\,dx \\ &= -\frac{1}{h^2}\ln\left(\frac{1+h(i+1)}{1+ih}\right) \end{aligned}$$

And finally, we have to compute the terms of $\mathbf{f}$.

$$\begin{aligned}
f_i &= \langle f, \phi_i \rangle \\
&= \int_0^1 x\phi_i(x)\,dx \\
&= \int_{h(i-1)}^{ih} x(x-h(i-1))/h\,dx + \int_{ih}^{h(i+1)} x(h(i+1)-x)/h\,dx \\
&= \frac{1}{h}\int_{h(i-1)}^{ih} x^2 - h(i-1)x\,dx + \frac{1}{h}\int_{ih}^{h(i+1)} h(i+1)x - x^2\,dx \\
&= \frac{1}{h}\left[\frac{1}{3}x^3 - \frac{h(i-1)}{2}x^2\right]_{h(i-1)}^{ih} + \frac{1}{h}\left[\frac{h(i+1)}{2}x^2 - \frac{1}{3}x^3\right]_{ih}^{h(i+1)} \\
&= \frac{1}{h}\left[\frac{1}{3}(ih)^3 - \frac{1}{2}h(i-1)(ih)^2 - \frac{1}{3}(h(i-1))^3 + \frac{1}{2}(h(i-1))^3\right] \\
&\quad + \frac{1}{h}\left[\frac{1}{2}(h(i+1))^3 - \frac{1}{3}(h(i+1))^3 - \frac{1}{2}h(i+1)(ih)^2 + \frac{1}{3}(ih)^3\right] \\
&= \frac{1}{h}\left[\frac{2}{3}(ih)^3 + \frac{1}{6}(h(i-1))^3 + \frac{1}{6}(h(i+1))^3 - i^3h^3\right] \\
&= h^2\left[\frac{1}{6}(i-1)^3 + \frac{1}{6}(i+1)^3 - \frac{1}{3}i^3\right] \\
&= h^2 i
\end{aligned}$$

(Please note: I do not expect you to simplify it this far. As long as you set up and compute the integrals correctly, that is sufficient.)

(e) Describe how $\mathbf{u}$, the solution of the system $\mathbf{K}\mathbf{u} = \mathbf{f}$, approximates the function u, the solution to the boundary value problem. In particular, how can you compare them to each other?

Solution. The important thing here is to remember that $\mathbf{u}$ is the vector for the coefficients of the function

$$v_n(x) = \sum_{k=1}^{n} u_k \phi_k(x),$$

and v_n is the best approximation to u in the subspace formed by the n hat functions. We would generally want to compare them by taking their difference and calculating the L^2 norm of this difference. Alternately, if you sample v_n and u at many points, you can then find their Euclidean distance, although this would be a less accurate way of telling how good your approximation is.