

CAAM 336
DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 1

SOLUTIONS

1. [20 points]

(a) Determine if the following sets are vector spaces or not- justify your answer. (Define addition and scalar multiplication in the obvious way.)

- i. The set of all vectors in \mathbb{R}^3 , (x_1, x_2, x_3) , such that $\|x\| = 1$.
- ii. The set of all vectors in \mathbb{R}^2 , (x_1, x_2) , such that $x_1 - 3x_2 = 0$.
- iii. $\{f \in C^2[0, 1] : -\frac{d^2f}{dx^2} + 4f = 0\}$
- iv. $\{f \in C[0, 1] : f(0) = f(1)\}$

Solution.

- i. This is not a vector space. The easiest way to see this is to take a vector \mathbf{x} where $\|\mathbf{x}\| = 1$ and multiply it by a scalar a . We then have $\|a\mathbf{x}\| = a\|\mathbf{x}\| = a \neq 1$.
 - ii. This is a vector space. Take two vectors $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ where $x_1 - 3x_2 = 0$ and $y_1 - 3y_2 = 0$. First check scalar multiplication: consider the vector $a\mathbf{x}$ for some constant a . Then $ax_1 - 3ax_2 = a(x_1 - 3x_2) = a \cdot 0 = 0$. Now check vector addition: consider the vector $\mathbf{x} + \mathbf{y}$. In the equation we have $x_1 + y_1 - 3(x_2 + y_2) = x_1 + y_1 - 3x_2 - 3y_2 = x_1 - 3x_2 + y_1 - 3y_2 = 0$.
 - iii. This is a vector space. Take f, g in the set, and a constant a . For scalar addition we have $-\frac{d^2af}{dx^2} + 4af = a(-\frac{d^2f}{dx^2} + 4f) = 0$. And for vector addition we have $-\frac{d^2(f+g)}{dx^2} + 4(f+g) = -\frac{d^2f}{dx^2} + 4f - \frac{d^2g}{dx^2} + 4g = 0$.
 - iv. This is also a vector space. As above, take functions f, g in the set, and a constant a . We already know that $f(0) = f(1)$, so then $af(0) = af(1)$. In the same manner, $f(0) + g(0) = f(1) + g(1)$.
- (b) Determine whether each of the following functions $\langle \cdot, \cdot \rangle$ determines an inner product on the vector space V . If not, demonstrate at least one property of the inner product which is violated.

i. $V = C^2[0, 1]$,

$$\langle u, v \rangle = \int_0^1 (1 + x^2)u'(x)v'(x)dx$$

ii. $V = C_D^2[0, 1] = \{v \in C^2[0, 1] \text{ such that } v(0) = v(1) = 0\}$,

$$\langle u, v \rangle = \int_0^1 (1 + x^2)u'(x)v'(x)dx$$

iii. $V = \mathbb{P}_2$, the set of polynomials of degree ≤ 2 ,

$$\langle a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0 \rangle = a_2b_2 + a_1b_1 + a_0b_0$$

Solution.

- i. This is not an inner product. Recall the positive definiteness property of inner products: $\langle u, u \rangle \geq 0$, and $\langle u, u \rangle = 0$ ONLY WHEN $u = 0$. So here's the problem: if we let $u = a$, some nonzero constant, then $\langle u, u \rangle = 0$, but $u \neq 0$. Not an inner product.
 - ii. This is an inner product. This is almost the same as the one above, except that the little problem of allowing nonzero constants has been removed. So everything is wonderful.
 - iii. This is also an inner product. In fact, this is the exact same vector space and inner product as \mathbb{R}^3 , only written in a strange way.
- (c) Determine if the following operators are linear or not. If they are not, justify your answer.

i. $L : C^2[0, 1] \rightarrow C[0, 1]$ defined by

$$Lu := -\frac{d}{dx} \left((1 + x^2) \frac{du}{dx} \right)$$

ii. $L : C^2[0, 1] \rightarrow C[0, 1]$ defined by

$$Lu := -\frac{d^2u}{dx^2}$$

Solution.

- i. This operator is linear. In other words $L(au + bv) = aLu + bLv$ for functions u and v and scalars a and b .
 - ii. This is not linear. Easiest way to see this is with scalar multiplication. If a is a scalar, observe that $L(au) = a^2Lu$, which is bad if you were hoping for linearity. Similar badness ensues if you try to have vector multiplication.
- (d) Define the operator $L : C^2[0, 1] \rightarrow C[0, 1]$ by

$$Lu := -\frac{d^2u}{dx^2}.$$

Find the null space of L , $N(L)$.

Solution. Recall that the null space of an operator is the set of all vectors u where $Lu = 0$. For our good friend the second derivative operator, the null space is all functions of the form $u(x) = ax + b$. Take two derivatives, and you've got zero.

2. [20 points]

Consider the matrix and vector

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -2 & 0 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

- (a) Compute the eigenvalues and eigenvectors of \mathbf{A} .
- (b) Solve $\mathbf{Ax} = \mathbf{b}$ by using the spectral method.

Solution. Okay, this one got a little screwy. In an effort to make the eigenvalues and eigenvectors easy to compute, the eigenvectors ended up not being orthogonal. This doesn't mesh with the way we've been doing the spectral method in class. As a result, part *b* did not count for much. However, I will show you how it could be done *almost* like the spectral method you've seen before.

- (a) First things first, we must calculate the characteristic polynomial.

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= \det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ -2 & -1 - \lambda & 2 \\ -2 & 0 & 3 - \lambda \end{pmatrix} \\ &= (1 - \lambda) \det \begin{pmatrix} -1 - \lambda & 2 \\ 0 & 3 - \lambda \end{pmatrix} \\ &= (1 - \lambda)(-1 - \lambda)(3 - \lambda) \end{aligned}$$

So the roots of the characteristic polynomial are $\lambda = 1, -1$, and 3 .

Now we must calculate the eigenvectors. Recall that the eigenvectors are the nullspace of $\mathbf{A} - \lambda\mathbf{I}$.

For $\lambda_1 = 1$, we want to find a vector where

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -2 & 2 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first row tells us nothing. The last row tells us that $x_1 = x_3$. With that, we conclude that $x_2 = 0$. So our first eigenvector is $\mathbf{u}_1 = (1, 0, 1)^T$.

On to $\lambda_2 = -1$.

$$\begin{pmatrix} 2 & 0 & 0 \\ -2 & 0 & 2 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This time the first row tells us that $x_1 = 0$. Knowing that, either of the other rows tells us that $x_3 = 0$. This leaves x_2 free to be anything. So keep it simple. The second eigenvector is $\mathbf{u}_2 = (0, 1, 0)^T$.

And finally, $\lambda_3 = 3$.

$$\begin{pmatrix} -2 & 0 & 0 \\ -2 & -4 & 2 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The first and third rows both tell us that $x_1 = 0$. The second row tells us that $2x_2 = x_3$. So once again keeping things simple, the third eigenvector is $\mathbf{u}_3 = (0, 1, 2)$.

- (b) Okay, now that that's done, let's solve this equation. The goal of the spectral method is to write $\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$ and use that to solve the equation. Plugging this into the equation gives

$$\begin{aligned} \mathbf{Ax} &= \mathbf{A}(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3) \\ &= a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \\ &= \mathbf{b}. \end{aligned}$$

At this point, we have a couple of options. One option is to treat this as a problem similar to the projection problems, and form the Gramm matrix. First take the inner product of everything with u_1 :

$$\begin{aligned} \langle \mathbf{u}_1, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= 2a_1 + 0a_2 + 6a_3 \\ &= \langle \mathbf{u}_1, \mathbf{b} \rangle \\ &= 3. \end{aligned}$$

Now with u_2 :

$$\begin{aligned} \langle \mathbf{u}_2, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= 0a_1 - 1a_2 + 3a_3 \\ &= \langle \mathbf{u}_2, \mathbf{b} \rangle \\ &= 1. \end{aligned}$$

Now finally, the inner product with u_3 :

$$\begin{aligned} \langle \mathbf{u}_3, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= 2a_1 - 1a_2 + 15a_3 \\ &= \langle \mathbf{u}_3, \mathbf{b} \rangle \\ &= 5. \end{aligned}$$

This forms the matrix equation

$$\begin{pmatrix} 2 & 0 & 6 \\ 0 & -1 & 3 \\ 2 & -1 & 15 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}.$$

All you need to do now is solve this equation. However, looking at this, you may think that this is no better than the original equation. In fact, it looks like it might be worse. Nobody did this anyway.

So I'm going to do something else. We have the equation

$$a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 = \mathbf{b}.$$

Now, recall the different eigenvectors:

$$\begin{aligned}\mathbf{u}_1 &= (1, 0, 1)^T \\ \mathbf{u}_2 &= (0, 1, 0)^T \\ \mathbf{u}_3 &= (0, 1, 2)^T.\end{aligned}$$

When I see that, I make an observation: \mathbf{u}_1 and \mathbf{u}_2 are orthogonal, and I can think of a simple vector that is orthogonal to both of them, the vector $\mathbf{v} = (1, 0, -1)^T$. So let's see what happens when we take the inner product of the equation with *this* vector.

$$\begin{aligned}\langle \mathbf{v}, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= 0a_1 + 0a_2 - 6a_3 \\ &= \langle \mathbf{v}, \mathbf{b} \rangle \\ &= -1.\end{aligned}$$

And look at that. With little effort at all, we have just found that $a_3 = 1/6$. Let's do this again.

Look at the last two eigenvectors. They may not be orthogonal to each other, but I can easily think of a vector which is orthogonal to both of them, the vector $\mathbf{v} = (1, 0, 0)^T$. Let's take the inner product with this one.

$$\begin{aligned}\langle \mathbf{v}, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= a_1 + 0a_2 + 0a_3 \\ &= \langle \mathbf{v}, \mathbf{b} \rangle \\ &= 1.\end{aligned}$$

That was even easier. Now we know that $a_1 = 1$.

To get a_2 , we can take the inner product of the equation with something that is orthogonal to both \mathbf{u}_1 and \mathbf{u}_3 . Looking at it for a minute, it seems that $\mathbf{v} = (1, 2, -1)^T$ works. So taking this last inner product, we get

$$\begin{aligned}\langle \mathbf{v}, a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2 + a_3\lambda_3\mathbf{u}_3 \rangle &= 0a_1 - 2a_2 + 0a_3 \\ &= \langle \mathbf{v}, \mathbf{b} \rangle \\ &= 1.\end{aligned}$$

And we have that $a_2 = -1/2$. Alternately, we could have used the equation obtained above saying that $-a_2 + 3a_3 = 1$ and the fact that we know a_3 to get this one.

With all of this, we have calculated \mathbf{x} :

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 1 \\ -\frac{1}{3} \\ \frac{4}{3} \end{pmatrix}}\end{aligned}$$

3. [20 points]

Consider the function $f(x) = \sqrt{x}$ on the interval $[0, 1]$.

- (a) Using the inner product $\langle u, v \rangle = \int_0^1 u(x)v(x) dx$, find a constant a so that $\{1, x + a\}$ is an orthogonal basis for \mathbb{P}_1 , the set of all polynomials of degree less than or equal to 1.
- (b) With this orthogonal basis for \mathbb{P}_1 , find the best approximation to f from \mathbb{P}_1 . (If you had trouble finding a in the previous part, just use it as an unknown in this part.)
- (c) Now, sample f at the points $x_1 = 0$, $x_2 = 1/4$, and $x_3 = 1$. Find the relationship $y = ax + b$ that best fits this data. You may want to use the fact that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Solution.

(a) We want to find a so that

$$\begin{aligned} 0 &= \langle 1, x + a \rangle \\ &= \int_0^1 x + a \, dx \\ &= \left[\frac{1}{2}x^2 + ax \right]_0^1 \\ &= \frac{1}{2} + a. \end{aligned}$$

In other words,

$$\boxed{a = -\frac{1}{2}}.$$

(b) Recall from the projection theorem that if we have an orthogonal basis, then

$$\text{proj}_{\mathbb{P}_1} f = \sum_{i=1}^n \frac{\langle w_i, f \rangle}{\langle w_i, w_i \rangle} w_i.$$

So we just need to calculate these inner products, and plug them into the formula.

$$\begin{aligned}
 \langle 1, \sqrt{x} \rangle &= \int_0^1 x^{1/2} dx \\
 &= \left[\frac{2}{3} x^{3/2} \right]_0^1 \\
 &= \frac{2}{3}, \\
 \left\langle x - \frac{1}{2}, \sqrt{x} \right\rangle &= \int_0^1 x^{3/2} - \frac{1}{2} x^{1/2} dx \\
 &= \left[\frac{2}{5} x^{5/2} - \frac{1}{3} x^{3/2} \right]_0^1 \\
 &= \frac{2}{5} - \frac{1}{3} \\
 &= \frac{1}{15}, \\
 \langle 1, 1 \rangle &= \int_0^1 1 dx \\
 &= 1, \\
 \left\langle x - \frac{1}{2}, x - \frac{1}{2} \right\rangle &= \int_0^1 \left(x - \frac{1}{2} \right)^2 dx \\
 &= \left[\frac{1}{3} \left(x - \frac{1}{2} \right)^3 \right]_0^1 \\
 &= \frac{1}{3} \left[\frac{1}{8} - \left(-\frac{1}{8} \right) \right] \\
 &= \frac{1}{12}.
 \end{aligned}$$

This then gives us the answer:

$$\boxed{\text{proj}_{\mathbb{P}_1} f = \frac{2}{3} + \frac{12}{15}x}.$$

- (c) We are once again finding a linear approximation to \sqrt{x} , but this time in a different way. There are three vectors here. First, the vector we get from sampling $f(x)$ at the three points. I will call this vector \mathbf{v} .

$$\mathbf{v} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

The other two vectors are the basis vectors representing x and 1, sampled at the three

points 0, 1/4, and 1. I will call them \mathbf{w}_1 and \mathbf{w}_2 .

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ \frac{1}{4} \\ 1 \end{pmatrix},$$

$$\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Returning to the projection theorem, recall that

$$\text{proj}_W \mathbf{v} = \sum_{i=1}^n x_i \mathbf{w}_i,$$

where $\mathbf{G}\mathbf{x} = \mathbf{b}$, $G_{ij} = \langle \mathbf{w}_j, \mathbf{w}_i \rangle$, $b_i = \langle \mathbf{w}_i, \mathbf{v} \rangle$. So now we need to solve for \mathbf{x} . To that end,

$$\mathbf{G} = \begin{pmatrix} \langle \mathbf{w}_1, \mathbf{w}_1 \rangle & \langle \mathbf{w}_2, \mathbf{w}_1 \rangle \\ \langle \mathbf{w}_1, \mathbf{w}_2 \rangle & \langle \mathbf{w}_2, \mathbf{w}_2 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 17/16 & 5/4 \\ 5/4 & 3 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} \langle \mathbf{w}_1, \mathbf{v} \rangle \\ \langle \mathbf{w}_2, \mathbf{v} \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 9/8 \\ 3/2 \end{pmatrix}$$

With this calculated, and the fact that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we can now find \mathbf{x} .

$$\mathbf{x} = \begin{pmatrix} 17/16 & 5/4 \\ 5/4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 9/8 \\ 3/2 \end{pmatrix}$$

$$= \frac{1}{51/16 - 25/16} \begin{pmatrix} 3 & -5/4 \\ -5/4 & 17/16 \end{pmatrix} \begin{pmatrix} 9/8 \\ 3/2 \end{pmatrix}$$

$$= \frac{16}{26} \begin{pmatrix} 27/8 - 15/8 \\ -45/32 + 51/32 \end{pmatrix}$$

$$= \frac{8}{13} \begin{pmatrix} 3/2 \\ 3/16 \end{pmatrix}$$

$$= \begin{pmatrix} 12/13 \\ 3/26 \end{pmatrix}$$

This then gives the approximation

$$y = \frac{12}{13}x + \frac{3}{26}.$$

4. [20 points]

Solve the boundary value problem

$$-\frac{d^2u}{dx^2} = 1 + x$$

$$u(0) = 2 \quad \frac{du}{dx}(1) = 1.$$

by using Fourier series, shifting the data if necessary.

Solution. First things first, this equation has inhomogeneous boundary conditions. So we must shift it. Define the function

$$p(x) = x + 2.$$

Observe that $p(0) = 2$ and $p'(1) = 1$. So now we define $v(x) = u(x) - p(x)$, and this new function satisfies the differential equation

$$-\frac{d^2v}{dx^2} = 1 + x$$

$$v(0) = 0 = \frac{dv}{dx}(1).$$

We will now solve for v using Fourier series. As we have seen in class, the eigenvalues of this operator are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4},$$

and the eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{(2n-1)\pi}{2}x\right).$$

The eigenvalues are evaluated a quarter off the integers because of the derivative at $x = 1$. The eigenfunctions are sines to ensure that they are zero at $x = 0$. The solution will be written as

$$v(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2}x\right).$$

Plug this in to the equation to get

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2\pi^2}{4} c_n \sin\left(\frac{(2n-1)\pi}{2}x\right) = 1 + x.$$

Now we take the inner product of both sides with the k^{th} eigenfunction (exploiting orthogo-

nality):

$$\begin{aligned}
 \int_0^1 \frac{(2k-1)^2 \pi^2}{4} c_k \sin^2 \left(\frac{(2k-1)\pi}{2} x \right) dx &= \int_0^1 (1+x) \sin \left(\frac{(2k-1)\pi}{2} x \right) dx \\
 \frac{(2k-1)^2 \pi^2}{8} c_k &= \left[-(1+x) \frac{2}{(2k-1)\pi} \cos \left(\frac{(2k-1)\pi}{2} x \right) \right]_0^1 \\
 &\quad + \int_0^1 \frac{2}{(2k-1)\pi} \cos \left(\frac{(2k-1)\pi}{2} x \right) dx \\
 &= \frac{2}{(2k-1)\pi} \\
 &\quad + \left[\frac{4}{(2k-1)^2 \pi^2} \sin \left(\frac{(2k-1)\pi}{2} x \right) \right]_0^1 \\
 &= \frac{2}{(2k-1)\pi} + \frac{4(-1)^{k+1}}{(2k-1)^2 \pi^2}
 \end{aligned}$$

We then have that

$$c_k = \frac{16}{(2k-1)^3 \pi^3} + \frac{32(-1)^{k+1}}{(2k-1)^4 \pi^4}.$$

So then

$$v(x) = \sum_{n=1}^{\infty} \left(\frac{16}{(2n-1)^3 \pi^3} + \frac{32(-1)^{n+1}}{(2n-1)^4 \pi^4} \right) \sin \left(\frac{(2n-1)\pi}{2} x \right).$$

But this isn't quite the answer yet. Don't forget, we want $u(x)$, and $u(x) = v(x) + p(x)$. In other words,

$$u(x) = x + 2 + \sum_{n=1}^{\infty} \left(\frac{16}{(2n-1)^3 \pi^3} + \frac{32(-1)^{n+1}}{(2n-1)^4 \pi^4} \right) \sin \left(\frac{(2n-1)\pi}{2} x \right).$$

5. [20 points]

Consider the equation,

$$-\frac{d}{dx} \left(\frac{1}{(1+x)^2} \frac{du}{dx}(x) \right) = 1, \quad 0 < x < 2,$$

with boundary conditions

$$u(0) = u(2) = 0,$$

- Find the analytic solution to the differential equation.
- Write down the weak form of the equation, that is,

$$a(u, v) = \langle f, v \rangle, \quad \text{for all } v \in C_D^2[0, 2],$$

in integral form. Show that the function $a(u, v)$ is an inner product on $C_D^2[0, 2]$.

- Suppose we take for ϕ_1, \dots, ϕ_N the standard piecewise linear 'hat' functions on the uniform mesh $h = 2/(N+1)$, $x_k = kh$,

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k]; \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}]; \\ 0, & \text{otherwise.} \end{cases}$$

Set up (but do not solve) the system $\mathbf{K}\mathbf{u} = \mathbf{f}$. Specifically, calculate \mathbf{K} and \mathbf{f} . You can either calculate them in general or for $N = 4$, whichever you like.

- (d) What is the relationship between \mathbf{u} , the solution of the system $\mathbf{K}\mathbf{u} = \mathbf{f}$ and u , the solution of the differential equation? In particular, how would you compare them to each other?

Solution.

- (a) After an indefinite integral, the equation becomes

$$\frac{1}{(1+x)^2} \frac{du}{dx}(x) = -x + C.$$

Multiply both sides by $(1+x)^2$. This gives

$$\frac{du}{dx}(x) = (-x + C)(1+x)^2.$$

We need to integrate again. The way it stands it looks like we need to multiply this out. However, I don't really like multiplying cubics. So I'm going to do a little algebra to make things easier on myself. Observe that

$$(-x + C)(1+x)^2 = -(1+x)^3 + (1+C)(1+x)^2.$$

If you like you can do the old trick of renaming the constant so that it doesn't have that extra +1 hanging around, but it's not necessary. Integrating this gives

$$u(x) = \frac{-1}{4}(1+x)^4 + \frac{1+C}{3}(1+x)^3 + D.$$

We need to solve for the constants, using $u(0) = 0 = u(2)$. Plugging this into the equation gives

$$\begin{aligned} 0 &= \frac{-1}{4} + \frac{1}{3}(1+C) + D \\ 0 &= \frac{-27}{4} + 9(1+C) + D. \end{aligned}$$

Subtracting these equations from each other gives

$$0 = -7 + \frac{26}{3}(1+C).$$

So then $(1+C) = 21/26$. Plug this into the first equation to get $D = -1/52$. So then the solution is

$$u(x) = \frac{-1}{4}(1+x)^4 + \frac{7}{26}(1+x)^3 - \frac{1}{52}.$$

- (b) The weak form of the differential equation is

$$\int_0^2 \frac{1}{(1+x)^2} u'(x) v'(x) dx = \int_0^2 v(x) dx,$$

for all $v \in C_D^2[0, 2]$. It is easy to see that $a(u, v) = a(v, u)$. Linearity of the inner product follows from the linearity of the derivative and the integral. Positive definiteness follows from the fact that in $C_D^2[0, 2]$, the only constant function is the zero function, so $u'(x)$ is never zero for functions in this vector space. So since $1/(1+x)^2 > 0$, $a(u, u) > 0$ for all nonzero functions in $C_D^2[0, 2]$, and is zero for the zero function.

- (c) Recall that \mathbf{K} is a matrix, where $K_{ij} = a(\phi_i, \phi_j)$. Also \mathbf{f} is a vector whose components are $f_i = \langle \phi_i, f \rangle$. We have dealt with the hat functions before, so we know that the only nonzero inner products involving hat functions with themselves are where the index is the same ($i = j$) and where the indices differ by one ($i = j - 1$). So let's compute these:

$$\begin{aligned} a(\phi_i, \phi_i) &= \int_0^2 \frac{1}{(1+x)^2} \phi_i'(x)^2 dx \\ &= \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} \frac{1}{(1+x)^2} dx \\ &= \frac{1}{h^2} \int_{(i-1)h}^{(i+1)h} \frac{1}{(1+x)^2} dx \\ &= \frac{1}{h^2} \left[\frac{-1}{1+x} \right]_{(i-1)h}^{(i+1)h} \\ &= \frac{1}{h^2} \left[\frac{1}{1+(i-1)h} - \frac{1}{1+(i+1)h} \right] \\ &= \frac{2}{h(1+ih)^2 - h^3}, \\ a(\phi_i, \phi_{i+1}) &= \int_0^2 \frac{1}{(1+x)^2} \phi_i'(x) \phi_{i+1}'(x) dx \\ &= \frac{-1}{h^2} \int_{ih}^{(i+1)h} \frac{1}{(1+x)^2} dx \\ &= \frac{1}{h^2} \left[\frac{1}{1+x} \right]_{ih}^{(i+1)h} \\ &= \frac{1}{h^2} \left[\frac{1}{1+(i+1)h} - \frac{1}{1+ih} \right] \\ &= \frac{-1}{h(1+ih)(1+(i+1)h)} \end{aligned}$$

This gives us \mathbf{K} .

$$\mathbf{K} = \frac{1}{h} \begin{pmatrix} \frac{2}{1+2h} & \frac{-1}{(1+h)(1+2h)} & 0 & \cdots & 0 \\ \frac{-1}{(1+h)(1+2h)} & \frac{2}{(1+h)(1+3h)} & \frac{-1}{(1+2h)(1+3h)} & 0 & \vdots \\ 0 & \frac{-1}{(1+2h)(1+3h)} & \frac{2}{(1+2h)(1+4h)} & \frac{-1}{(1+3h)(1+4h)} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \frac{-1}{(1+(N-1)h)(1+Nh)} & \frac{2}{(1+(N-1)h)(1+(N+1)h)} \end{pmatrix}$$

Now we compute \mathbf{f} .

$$\begin{aligned}
 \langle \phi_i, f \rangle &= \int_0^2 \phi_i(x) dx \\
 &= \frac{1}{h} \int_{(i-1)h}^{ih} x - (i-1)h dx \\
 &\quad + \frac{1}{h} \int_{ih}^{(i+1)h} (i+1)h - x dx \\
 &= \frac{1}{h} \left[\frac{1}{2}x^2 - (i-1)hx \right]_{(i-1)h}^{ih} \\
 &\quad + \frac{1}{h} \left[(i+1)hx - \frac{1}{2}x^2 \right]_{ih}^{(i+1)h} \\
 &= \frac{1}{h} \left[\frac{1}{2}(ih)^2 - (i-1)ih^2 - \frac{1}{2}(i-1)^2h^2 + (i-1)^2h^2 \right] \\
 &\quad + \frac{1}{h} \left[(i+1)^2h^2 - \frac{1}{2}(i+1)^2h^2 - (i+1)ih^2 + \frac{1}{2}(ih)^2 \right] \\
 &= i^2h + \frac{1}{2}(i-1)^2h + \frac{1}{2}(i+1)^2h \\
 &= (i+1)^2h
 \end{aligned}$$

- (d) The important thing here is to remember that \mathbf{u} is the vector for the coefficients of the function

$$v_n(x) = \sum_{k=1}^n u_k \phi_k(x).$$

And v_n is the best approximation to u in the subspace formed by the n hat functions. We would generally want to compare them by taking their difference and calculating the L^2 norm of this difference. Alternately, if you sample v_n and u at many points, you can then find their Euclidean distance, although this would be a less accurate way of telling how good your approximation is.