CAAM 336
DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 1

Posted 3 March 2006.

Due 5pm on Wednesday, 8 March 2006.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [20 points]
   Consider the matrix and vector

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Compute the eigenvalues and eigenvectors of $\mathbf{A}$.

   (b) Show that the matrix

$$\mathbf{A}_\alpha = \mathbf{A} + \alpha\mathbf{I} = \begin{pmatrix} \alpha & 0 & 2 \\ 0 & \alpha & 0 \\ 2 & 0 & \alpha \end{pmatrix}.$$

   has the same eigenvectors as $\mathbf{A}$. What are the eigenvalues of $\mathbf{A}_\alpha$?

   (c) Solve $\mathbf{A}_\alpha \mathbf{x}_\alpha = \mathbf{b}$ for $\mathbf{x}_\alpha$ using the spectral method.
   For which values of α does this procedure fail?
   What does this imply about the existence of $\mathbf{x}_\alpha$ in those cases?

2. [25 points]
Let $C^2_D[0,\pi] = \{u \in C^2[0,\pi] : u(0) = u(\pi) = 0\}$.

(a) Derive the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ and eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ of the symmetric linear operator $L : C^2_D[0,\pi] \to C[0,\pi]$ defined by

$$Lu = -\frac{d^2u}{dx^2}, \qquad 0 < x < \pi.$$

(b) Using Problem 1 as a hint, write down the eigenvalues and eigenfunctions of the operator $L_\alpha : C^2_D[0,\pi] \to C[0,\pi]$

$$L_\alpha u \;=\; \Big(-\frac{d^2}{dx^2} + \alpha\Big)u \;=\; -\frac{d^2u}{dx^2} + \alpha u, \qquad 0 < x < \pi.$$

(c) Describe how to use the spectral method to solve the differential equation

$$-\frac{d^2u}{dx^2} + \alpha u(x) = f(x), \qquad 0 < x < \pi,$$

with boundary conditions $u(0) = u(\pi) = 0$.

(d) Use the spectral method to solve the differential equation

$$-\frac{d^2u}{dx^2} + u(x) = \sin(7x), \qquad 0 < x < \pi,$$

with $u(0) = u(\pi) = 0$.

(Hint: The eigenfunctions of L_α associated with distinct eigenvalues are orthogonal.)

3. [25 points]

(a) Let V be a vector space endowed with an inner product $(\cdot,\cdot)$, and suppose that V_N is some finite-dimensional subspace of V:

$$V_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\},$$

where the vectors $\phi_1, \ldots, \phi_N$ are linearly independent.

Given some $v \in V$, describe in detail how you would construct the $v_N \in V_N$ that best approximates v.

(b) Does the best approximation $v_N \in V_N$ constructed in part (a) depend upon the basis that we use for V_N? That is, suppose we have some different set of linearly independent vectors $\psi_1, \ldots, \psi_N$ such that

$$\begin{aligned} V_N &= \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\} \\ &= \text{span}\{\psi_1, \psi_2, \ldots, \psi_N\}. \end{aligned}$$

Do we obtain a different best approximation if we use $\psi_1, \ldots, \psi_N$ instead of $\phi_1, \ldots, \phi_N$? Explain.

(c) Does the best approximation v_N constructed in part (a) depend upon the choice of the inner product? That is, if we replace $(\cdot,\cdot)$ with some different inner product $\langle\cdot,\cdot\rangle$, will the best approximation change? Explain.

(d) Consider the following inner product defined on $V = C^1[0,1]$, the set of continuous functions on $[0,1]$ with a continuous first derivative:

$$(u,v) = \int_0^1 u(x)v(x) + \Big(\frac{du}{dx}(x)\Big)\Big(\frac{dv}{dx}(x)\Big)\,dx.$$

Find the best approximation to $v(x) = e^x$ from the space

$$V_2 = \text{span}\{\phi_1, \phi_2\}, \qquad \phi_1(x) = 1, \quad \phi_2(x) = 1 - 2x.$$

(Please compute all integrals and solve the linear system for full credit.)

4. [30 points]
Consider the *Euler–Bernoulli beam equation*,

$$\frac{d^2}{dx^2}\Big(\kappa(x)\frac{d^2u}{dx^2}(x)\Big) = f(x), \qquad 0 < x < 1,$$

with boundary conditions describing a beam that is clamped at both ends:

$$u(0) = u(1) = 0, \qquad \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0.$$

Here $\kappa(x)$ is a positive-valued function that describes the material properties of the beam.

> **NOTE**: If you cannot solve the problems below for the beam equation, you may substitute the more familiar equation
>
> $$-\frac{d}{dx}\Big(\kappa(x)\frac{du}{dx}(x)\Big) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0$$
>
> on $C_D^2[0,1]$ with $\kappa(x) > 0$ for partial credit.

(a) State what it means for a linear operator to be *symmetric*, and show that the operator $L : C_D^4[0,1] \to C[0,1]$, defined by

$$Lu = \frac{d^2}{dx^2}\Big(\kappa(x)\frac{d^2u}{dx^2}(x)\Big), \qquad 0 < x < 1,$$

is a symmetric operator acting on the space

$$C_D^4[0,1] = \Big\{u \in C^4[0,1] : u(0) = u(1) = \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0\Big\}.$$

(b) The eigenvalues and eigenvectors of this operator are difficult to compute, so finite element approximations are particularly useful here. Derive the weak form of the beam equation with the above boundary conditions, i.e., derive the weak problem

$$a(u,v) = (f,v), \quad \text{for all } v \in C_D^4[0,1].$$

Specify the bilinear form $a(u,v)$, and show that it is an inner product on $C_D^4[0,1]$.

(c) Suppose that $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$ is an N-dimensional subspace of $C_D^4[0,1]$. (Do not assume a particular form for the functions $\phi_1, \ldots, \phi_N$ at this point.) Show how the Galerkin problem

$$a(u_N, v) = (f,v), \quad \text{for all } v \in V_N$$

leads to the linear system $\mathbf{Ku} = \mathbf{f}$. Be sure to specify the entries of $\mathbf{K}$, $\mathbf{u}$, and $\mathbf{f}$.

(d) Now suppose we take for $\phi_1, \ldots, \phi_N$ the standard piecewise linear 'hat' functions used on Problem Sets 6 and 7. Are these functions suitable for this problem? If so, describe the location of the nonzero entries of the matrix $\mathbf{K}$. If not, propose a better choice for the functions $\phi_1, \ldots, \phi_N$ and the describe which entries of $\mathbf{K}$ are nonzero for that choice.