CAAM 336
DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 2

Posted 8 December 2006.

Due 12 noon on Friday, Dec 15 2006.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

   ____

   ____

   ____

7. Staple this page to the front of your exam.

1. [40 points]
On the last homework assignment we considered the the wave equation posed on a square domain Ω in two dimensions with Dirichlet boundary conditions. For this problem you will analyze the wave equation on the square again, but with mixed boundary conditions:

$$\frac{\partial^2 u}{\partial t^2}(x, y, t) = \Delta u$$

for $t \geq 0$, $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with boundary conditions

$$u(x, 0, t) = u(x, 1, t) = u(0, y, t) = \frac{\partial u}{\partial x}(1, y, t) = 0$$

and initial conditions

$$u(x, y, 0) = \psi(x, y)$$

and

$$\frac{\partial u}{\partial t}(x, y, 0) = \gamma(x, y)$$

.

(a) [20 pts] Find the eigenvalues $\lambda_{j,k}$ and associated eigenfunctions $\phi_{j,k}$ for the corresponding steady state problem:

$$-\Delta\phi = \lambda\phi$$

$$\phi(x, 0) = \phi(x, 1) = \phi(0, y) = \frac{\partial \phi}{\partial x}(1, y) = 0$$

(b) [20 pts] We wish to write the solution to the wave equation in the form

$$u(x, y, t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\phi_{j,k}(x, y).$$

Derive an ordinary differential equation for each $a_{jk}(t)$. Write down the solution of this ordinary differential equation, and use this to write a formula for $u(x, y, t)$ that satisfies the initial conditions.

2. [25 points]
Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 1$ and $t \ge 0$ with homogeneous Neumann boundary conditions,

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$

and initial data

$$u(x,0) = \psi(x)$$

(a) Compute all the eigenvalues λ_k and eigenfunctions ϕ_k of the operator

$$Lu = -\frac{d^2 u}{dx^2}$$

for $0 \le x \le 1$ subject to the homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(1) = 0.$$

(b) Suppose we wish to write the solution to the heat equation in the form

$$u(x,t) = \sum_k a_k(t)\phi_k(x).$$

Write down an ordinary differential equation for the coefficients $a_k(t)$; be sure to specify the initial conditions for $a_k(0)$.

(c) Solve the ordinary differential equations from part (b). Please explain any special cases.

(d) What happens to the solution $u(x,t)$ as $t \to \infty$?

(e) Find the solution $u(x,t)$ when $u(x,0) = 3 + \cos \pi x - 2\cos 10\pi x$.

3. [20 points]
   Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -10 & 9 \\ 9 & -10 \end{pmatrix}.$$

   (a) Compute the matrix exponential $e^{t\mathbf{A}}$ for this matrix and describe its behavior as $t \to \infty$.

   (b) Write down the forward Euler and backward Euler methods for approximating the solution of the differential equation

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t).$$

   (c) For the matrix $\mathbf{A}$ given above, what is the largest value of the time step Δt for which the approximate solution produced by the forward Euler method will mimic the behavior of the true solution as $t \to \infty$?

   (d) Is there any restriction on Δt for the backward Euler solution to mimic the qualitative behavior of the true solution? If so, find it.

   (e) What is a disadvantage to using backward Euler?

4. [15 points]
Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t).$$

for $0 \le x \le 1$, $t \ge 0$, with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = \psi(x).$$

Describe how to solve this differential equation using the finite element method. Provide as much detail as you can about the following points.

(a) What is the weak form of this differential equation?

(b) State the Galerkin problem based on the approximating subspace $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$ for the piecewise linear basis functions $\phi_1, \ldots, \phi_N$.

(c) Show how this problem leads to a differential equation of the form

$$\mathbf{A}\frac{d}{dt}\mathbf{a}(t) + \mathbf{B}\mathbf{a}(t) = \mathbf{f},$$

where you should specify the entries of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{f}$. You may leave them in inner product or integral form.