# CAAM 336
# DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Final Exam

Posted 23 April 2008.

Due 5pm on Wednesday, 30 April 2008 for graduating seniors.

Due 5pm on Friday, 2 May 2008 for everyone else.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are three questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   *Show all your work to maximize partial credit.*
5. Print your name on the line below:

   _____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [30 points]
   Consider Poisson's equation:

$$\Delta u = 2$$

   for $0 \leq x \leq 2$ and $0 \leq y \leq 3$ with boundary conditions

$$u(x,0) = u(x,3) = u(0,y) = u(2,y) = 0$$

   (a) [10 pts] Find the eigenvalues $\lambda_{i,j}$ and associated eigenfunctions $\phi_{i,j}$ for the Laplacian with the same boundary conditions:

$$-\Delta\phi = \lambda\phi$$

$$\phi(x,0) = \phi(x,3) = \phi(0,y) = \phi(2,y) = 0.$$

   (b) [20 pts] Use these eigenfunctions and eigenvalues to find a series solutions to the Poisson equation above.

2. [30 points]
Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0$$

with initial conditions

$$\begin{aligned} u(x,0) &= 2 - x, \\ \frac{\partial u}{\partial t}(x,0) &= x^2 - 2x. \end{aligned}$$

(a) [10 points] Write down the solution to this equation on the whole real line $(-\infty < x < \infty)$

(b) [20 points] Now consider the differential equation with the same initial conditions, only now on the domain $0 \leq x \leq 1$ with boundary conditions

$$\begin{aligned} u(0,t) &= 1 \\ \frac{\partial u}{\partial x}(1,t) &= -1 \end{aligned}$$

Find the Fourier series solution to this initial boundary value problem.

3. [40 points]

Consider the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

for $x \in (0,1)$ and $t > 0$, with homogenous Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0,$$

and an initial condition

$$u(x,0) = \psi(x).$$

Describe how to solve this differential equation using the finite element method. Provide as much detail as you can about the following points.

(a) [8 points] What is the weak form of this differential equation?

(b) [8 points] State the Galerkin problem based on the approximating subspace $V_N = \text{span}\{\phi_1, \cdots, \phi_N\}$, where $\phi_j$ are the "hat functions" centered on the evenly spaced mesh points $x_j$ for $1 \leq j \leq N$.

(c) [8 points] Show how this problem leads to an initial-valued, ordinary differential equation of the form

$$\begin{aligned} \frac{d\mathbf{a}}{dt}(t) &= \mathbf{A}\mathbf{a}(t) \qquad t > 0 \\ \mathbf{a}(0) &= \mathbf{a}_0 \end{aligned}$$

for $\mathbf{a} : \mathbb{R} \to \mathbb{R}^n$.

i. What do the entries in the vector valued function $\mathbf{a}(t)$ represent?

ii. How do you derive the initial condition vector $\mathbf{a}_0$?

iii. Write down as explicitly as you can what the matrix $\mathbf{A}$ is in terms of the mass and stiffness matrix.

(d) [8 points] Write down a formula for the solution to the differential equation from part c. Describe, qualitatively, the behaviour of $\mathbf{a}(t)$ as time goes to infinity, given that all the eigenvalues of $\mathbf{A}$ are negative, and your reasoning for this description.

(e) [8 points] Write down the backward Euler rule for computing the approximations to $\mathbf{a}(t)$ at the $k+1$ time step, for time steps of size $\Delta t$. Describe how you would pick your time step $\Delta t$ to ensure that the approximation $\mathbf{a}_k$ behaves qualitatively similarly to $\mathbf{a}(t_k)$, given that all the eigenvalues of $\mathbf{A}$ are negative?