CAAM 336

DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 2

Posted Monday, April 23 2007.

Due 12 noon on Monday, May 7 2007.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points]
   For this problem you will solve the Poisson equation on the square, but with mixed boundary conditions:
   $$\Delta u = 1$$
   for $0 \le x \le 1$ and $0 \le y \le 1$ with boundary conditions
   $$u(x,0) = \frac{\partial u}{\partial y}(x,1) = \frac{\partial u}{\partial x}(0,y) = u(1,y) = 0.$$

   (a) [15 pts] Find the eigenvalues $\lambda_{j,k}$ and associated eigenfunctions $\phi_{j,k}$ for the Laplacian with the same boundary conditions:
   $$-\Delta\phi = \lambda\phi$$
   $$\phi(x,0) = \frac{\partial \phi}{\partial y}(x,1) = \frac{\partial \phi}{\partial x}(0,y) = \phi(1,y) = 0.$$

   (b) [10 pts] Use these eigenfunctions and eigenvalues to find a series solution to the Poisson equation above.

2. [25 points]
Find a Fourier series solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 2$ and $t \ge 0$ with inhomogeneous Dirichlet boundary conditions,

$$u(0,t) = 0, \quad u(2,t) = 3$$

and initial data

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = 0.$$

(A note: The initial and boundary conditions are not compatible here- just ignore this and proceed as if they are compatible.)

3. [25 points]
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -20 & -18 \\ -18 & -20 \end{pmatrix}.$$

(a) Compute the matrix exponential $e^{t\mathbf{A}}$ for this matrix and describe its behavior as $t \to \infty$.

(b) Write down the forward Euler and backward Euler methods for approximating the solution of the differential equation

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t).$$

(c) For the matrix $\mathbf{A}$ given above, what is the largest value of the time step Δt for which the approximate solution produced by the forward Euler method will mimic the behavior of the true solution as $t \to \infty$?

(d) Is there any restriction on Δt for the backward Euler solution to mimic the qualitative behavior of the true solution? If so, find it.

(e) What is a disadvantage to using backward Euler?

4. [25 points]
Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = x\cos t.$$

for $0 \le x \le 1$, $t \ge 0$, with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = x(1-x).$$

(a) What is the weak form of this differential equation?

(b) State the Galerkin problem based on the approximating subspace $V_3 = \text{span}\{\phi_1, \phi_2, \phi_3\}$ for the piecewise linear basis functions.

(c) Show how this problem leads to a differential equation of the form

$$\mathbf{A}\frac{d}{dt}\mathbf{a}(t) + \mathbf{B}\mathbf{a}(t) = \mathbf{f},$$

$$\mathbf{a}(0) = \mathbf{a}_0$$

Provide integral formulas (involving the ϕ_i's) for the entries of A, B, and f, and provide a reasonable initial vector $\mathbf{a}_0$. Do NOT evaluate the integrals.