CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 13

Posted Friday 11 April, 2008. Due Friday 18 April, 2006 in class.

1. [50 points]

   We wish to approximate the solution to the homogenous wave equation

   $$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 \le x \le 1,\ t \ge 0$$

   with homogeneous Dirichlet boundary conditions

   $$u(0,t) = u(1,t) = 0$$

   and initial conditions

   $$\begin{aligned} u(x,0) &= \psi(x) \\ \frac{\partial u}{\partial t}(x,0) &= \gamma(x) \end{aligned}$$

   using the finite element method (method of lines).

   a. Write down the weak formulation of the initial, boundary value vave problem by taking the $L^2(0,1)$ inner product of both sides with a test function $v(x) \in C^2_D(0,1)$. Integrate by parts to write the weak formulation in terms of the familiar energy inner product :

   $$a(u,v) = \int_0^1 c^2 \frac{\partial u}{\partial x} \frac{dv}{dx}\, dx$$

   b. Let $N \ge 1$, $h = 1/(N+1)$, and $x_k = kh$ for $k = 0, \ldots, N+1$. We define our basis of hat functions in the usual way

   $$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

   Let $V_N = \text{span}\{\phi_k\}_{k=1}^N$. Consider an approximate solution to the weak formulation with test functions in V_N to have the form

   $$u_N(x,t) = \sum_{k=1}^{N} a_k(t)\phi_k(x).$$

   Write down the system of second order, ordinary differential equations that determines the coefficients $a_k(t)$ for $k = 1, \cdots, N$. Specify the entries of the mass and stiffness matrix and the load vectors.

Note: We can write the ODE from part b in the following way

$$\frac{d^2\mathbf{a}}{dt^2} = \mathbf{A}\mathbf{a}(t)$$

where $\mathbf{A} = -c^2\mathbf{M}^{-1}\mathbf{K}$. We can write the above system of second order ODE's as a first order system of ODE's by defining the auxilliary function $\mathbf{b}(t)$ as

$$\frac{d\mathbf{a}}{dt} = \mathbf{b}.$$

Plugging this into the second order ODE gives the system

$$\left.\begin{array}{rcl} \dfrac{d\mathbf{a}}{dt}(t) & = & \mathbf{b}(t) \\ \dfrac{d\mathbf{b}}{dt}(t) & = & \mathbf{A}\mathbf{a}(t). \end{array}\right\} \iff \frac{d}{dt}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}(t) = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{A} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{a}(t)\\ \mathbf{b}(t)\end{bmatrix}$$

where our initial conditions are $\mathbf{a}(0)$, which come from sampling $\psi(x)$ at the mesh points, and $d\mathbf{a}(0)/dt = \mathbf{b}(0)$, which come from sampling $\gamma(x)$ at the mesh points. We can simplify the above by defining

$$\mathbf{v}(t) = \begin{bmatrix}\mathbf{a}(t)\\ \mathbf{b}(t)\end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{A} & \mathbf{0}\end{bmatrix}$$

Then the ODEs we are interested in can now be written as a single system of first order ODEs

$$\frac{d\mathbf{v}}{dt} = \mathbf{B}\mathbf{v}(t) \tag{1}$$

c. Let $\psi(x) = e^{100x(1-x)} - 1$ and $\gamma(x) = 0$, and $c = 1$. Use Backward Euler to approximate the solution to equation 1 and plot the approximate solution $u_N(x,t)$ at every tenth of a second for the first half second, i.e. from $t = .1$ to $t = .5$. Use $\Delta_t = .01$ and let $N = 100$ (Note that the coefficents to $u_N(x,t)$ are the first half of the vector $\mathbf{v}(t)$)

d. Let $\psi(x) = 0$ and $\gamma(x) = e^{100x(1-x)} - 1$ and $c = 1$ and repeat part c.

2. [50 points]
The goal of this exercise is to solve the wave equation on a square domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

with $0 \le x \le 1$, and $0 \le y \le 1$, and $t \ge 0$. Take homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

for all x and y such that $0 \le x \le 1$ and $0 \le y \le 1$ and all $t \ge 0$, and consider the initial conditions

$$u(x,y,0) = \psi(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\phi_{j,k}(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0) = \gamma(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} d_{j,k}\phi_{j,k}(x,y).$$

Here the $\phi_{j,k}(x,y) = \sin(j\pi x)\sin(k\pi y)$, $j,k \ge 1$ are the eigenfunctions of the operator

$$Lu = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

with homogeneous Dirichlet boundary conditions. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k} = (j^2 + k^2)\pi^2$ computed in class.

(a) We wish to write the solution to the wave equation in the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\phi_{j,k}(x,y).$$

Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

$$\frac{d^2 a_{j,k}}{dt^2}(t) = -\lambda_{j,k} a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0) = b_{j,k}, \qquad \frac{da_{j,k}}{dt}(0) = d_{j,k}.$$

(b) Write down the solution to the differential equation in part (a).

(c) Use your solution to part (b) to write out a formula for the solution $u(x, y, t)$.

(d) Suppose that the initial conditions are

$$\psi(x, y) = 300xy(1 - x)(1 - y)^2, \qquad \gamma(x, y) = 0.$$

Modify the `wave2d.m` program from the class web site to plot the solution as a function of time. Submit plots of your solution at times $t = 0$, $\sqrt{2}/2$, $\sqrt{2}$, $3\sqrt{2}/2$, $2\sqrt{2}$.
(Note that we have not normalized these eigenfunctions; $(\phi_{j,k}, \phi_{j,k}) = 1/4$. You will need to take this into account in your formula for $b_{j,k}$.)