# CAAM 336 · DIFFERENTIAL EQUATIONS

## Problem Set 5

Posted Wednesday 6 February 2008. Due Wednesday 13 February 2008, in class.

1. [20 points]
   Suppose that

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) Compute *by hand* the eigenvalues and eigenvectors of this matrix.

   (b) Verify that these eigenvectors are orthogonal.

   (c) Solve the linear system $\mathbf{Ax} = \mathbf{b}$ using the spectral method, where

$$\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

2. [30 points]
   We saw in class that the linear operator

$$Lu = -\frac{d^2}{dx^2}u$$

   acting on the vector space $C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$ has eigenvalues

$$\lambda_k(L) = k^2\pi^2, \qquad k = 1, 2, \ldots$$

   with corresponding eigenvectors

$$v_k(x) = \sin(k\pi x), \qquad k = 1, 2, \ldots.$$

   For this problem, we wish to see how well the eigenvalues of this operator are approximated by the eigenvalues of our matrix

$$\mathbf{A}_N = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \in R^{N\times N},$$

   where $h = 1/(N+1)$ and unspecified entries are zero.

   (a) Use the `eig` command in MATLAB to compute the eigenvalues of this matrix for $N = 64$. Produce a careful `semilogy` plot showing, on the horizontal axis, the index $k$ for $k = 1, \ldots N$ and, on the vertical axis, the error between the eigenvalues of $\mathbf{A}_N$ and the exact eigenvalues $\lambda_k(L) = k^2\pi^2$ for the operator $L$. Which eigenvalues of $L$ are most accurately approximated?

   (b) Now create a `loglog` plot showing how the error in the smallest four eigenvalues changes as $N = 8$, 16, 32, 64, 128. On the horizontal axis you should have $N$, and on the vertical axis you should have the error, i.e., $|\lambda_k(\mathbf{A}_N) - \lambda_k(L)|$. There should be one curve each for $k = 1, 2, 3, 4$.

(c) Comment (qualitatively) on how well the *eigenvectors* of $\mathbf{A}_N$ approximate those of $L$ for small values of $k$ and $N = 64$. You can compute eigenvalues in MATLAB using `[V,D] = eig(A)`. (Spare a tree: please *do not* print out all the entires of the matrix of eigenvectors!)

The following two problems use the inner product

$$(u, v) = \int_0^1 u(x)v(x)\,dx.$$

3. [20 points]
Consider the operator $L : C^1_\ell[0,1] \to C[0,1]$ defined by

$$Lu = \frac{du}{dx},$$

where

$$C^1_\ell[0,1] = \{u \in C^1[0,1], u(0) = 0\}.$$

(a) Show that $L$ is a linear operator.

(b) Is $L$ symmetric?

(c) Show that $L$ has *no* eigenvalues, that is, demonstrate that there exist no nonzero $u \in C^1_\ell[0,1]$ and $\lambda \in \mathbb{C}$ for which $Lu = \lambda u$.

4. [30 points]
Consider the linear operator $L_b : C^2_b[0,1] \to C[0,1]$ defined by

$$L_b u = -\frac{d^2u}{dx^2},$$

where

$$C^2_b[0,1] = \left\{u \in C^2[0,1] : \frac{du}{dx}(0) = u(1) = 0\right\}.$$

(a) Is $L_b$ symmetric?

(b) What is the null space of $L_b$?
That is, find all $u \in C^2_b[0,1]$ such that $L_b u(x) = 0$ for all $x \in [0,1]$.

(c) Show that $(L_b u, u) \geq 0$ for all nonzero $u \in C^2_b[0,1]$ and explain why this implies that $\lambda \geq 0$ for all eigenvalues $\lambda$.

(d) Find the eigenvalues and eigenfunctions of $L_b$.