1 Inner Products of Piecewise Defined Functions

The homework asks you to take the inner product of some piecewise linear functions. The integrals in theory aren't that bad. But setting them up properly can be a pain. So I'd like to do a similar example to illustrate how it works.

Consider the functions

$$f_n(x) = \begin{cases} (x-1-n)^2(x+1-n)^2 & -1+n \leq x \leq 1+n \\ 0 & \text{otherwise} \end{cases}$$

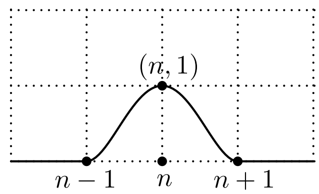

Figure 1: f_n, centered at n, max is 1

These functions are zero most places, and are only nonzero for an interval of length two centered at the value n. The important thing to notice is that if you grab two f_n's, sometimes the place where they're nonzero overlaps, and sometimes it doesn't.

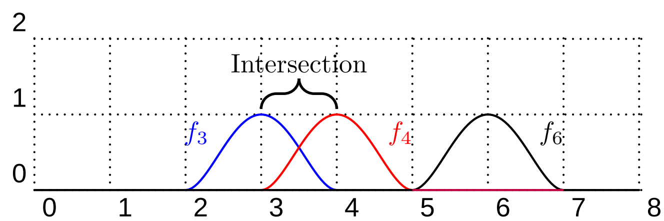

Figure 2: The functions f_3, f_4, and f_6. f_3 and f_4 are both nonzero between 3 and 4, but they are both zero where f_6 is nonzero.

The figure shows some examples of this. We are dealing with the L^2 inner product, that is,

$$\langle u, v \rangle = \int_{-\infty}^{\infty} u(x)v(x)\, dx.$$

(Whoops. I cheated and slipped in an infinity when you weren't looking. Don't worry, it's just for the sake of this example. And 'cause I feel like it.) So let's look at $\langle f_4, f_6 \rangle$. If you multiply these functions together, if one is nonzero, the other is zero, so $\langle f_4, f_6 \rangle = 0$. In fact, remember that f_n is nonzero between $n-1$ and $n+1$, so if you take the inner product of two of these functions and the difference between their n's is more than 1, the inner product will be zero.

So there are only two times when we're going to get a nonzero inner product: when the n's are the same, and when the n's are off by one.

I'll use f_3 and f_4, the ones illustrated in the figure, to show how this works. Consider $\langle f_3, f_4 \rangle$. Looking at the picture, we see that f_3 and f_4 are both nonzero between 3 and 4. So

$$\begin{aligned}\langle f_3, f_4 \rangle &= \int_3^4 f_3(x) f_4(x)\, dx \\ &= \int_3^4 (x-1-3)^2(x+1-3)^2(x-1-4)^2(x+1-4)^2\, dx \\ &= \int_3^4 (x-2)^2(x-3)^2(x-4)^2(x-5)^2\, dx\end{aligned}$$

And so we come to this terrible, terrible looking integral. Sure, it's a polynomial, but it's an eighth order polynomial. I don't know about you, but I don't want to do that. That's why we have access to things like Matlab, Maple, and Mathematica. Toss this integral in there, and we get

$$\langle f_3, f_4 \rangle = \frac{103}{630} \approx 0.1634920635,$$

just as I suspected. Or...not. I'm just glad I didn't compute that myself.

To finish this off, we should compute the inner product of f_3 with itself. The important thing to notice about this is that the place where f_3 is nonzero (commonly called its support) is the interval from 2 to 4. Note that this is two units long, whereas the common support of f_3 and f_4 was only one unit long. So here we go.

$$\begin{aligned}\langle f_3, f_4 \rangle &= \int_2^4 f_3(x)^2\, dx \\ &= \int_2^4 (x-2)^4(x-4)^2\, dx \\ &= \frac{128}{105} \approx 1.219047619\end{aligned}$$

The important thing to take away from this is that when dealing with inner products of functions that are zero most places, the first thing you need to do is check to see where both functions are nonzero, then just integrate there.

2 Using Maple to Find Eigenfunctions

It's possible you've used Maple to plot the occasional function, find the occasional integral, and target the occasional missile (okay, maybe not that last one). But not everyone knows you can use Maple to find eigenfunctions without too much difficulty. Let's say we're hanging out on the interval $[0, 1]$ and we want to find the Fourier sine coefficients of e^x. To do this we need to find the value of the integral

$$\int_0^1 e^x \sin(n\pi x)\, dx.$$

We could do this to old school way and integrate by parts a couple of times. Or we could do the guess and check method, my personal favorite. Or we could turn sin into complex exponentials (as some people in class say they have seen before). But let's say you're in a hurry, or are lazy, or maybe you just want to check your work. Let's go to Maple. If you type in:

```
int(exp(x)*sin(n*Pi*x),x=0..1);
```

(that capital p is important) you get

$$-\frac{-n\pi + en\pi\cos(n\pi) - e\sin(n\pi)}{1 + n^2\pi^2}.$$

Wait, didn't I just say that capital p was important? Sure, spelling "Pi" with a capital p tells Maple that I'm talking about the number π, and not the generic Greek letter π, but the answer it gave me is still a bit of a mess. So we need to tell Maple that n is an integer, since I don't think it took the hint. Type in

```
assume(n,integer);
```

Then go back and calculate that integral again. This time you get

$$-\frac{n\sim\pi(-1+(-1)^{n\sim}e)}{1+n\sim^2\pi^2}.$$

This looks nice. In fact, this looks just about right. Except for all those weird tildes everywhere. That is just Maple's way of letting you know that it thinks n is special. So if I were to write this down as an answer, I'd write it as

$$\frac{n\pi(1+(-1)^{n+1}e)}{1+n^2\pi^2}.$$

As long as you're using Maple, it thinks n is an integer, and now you're primed to find more coefficients. For example, using Maple this way I can quickly find that

$$\int_0^1 x\sin(x)\sin(n\pi x) = -n\pi\frac{2+2(-1)^{n+1}\cos(1)+n^2\pi^2\sin(1)(-1)^n+(-1)^{n+1}\sin(1)}{1-2n^2\pi^2+n^4\pi^4}.$$

Would you have wanted to do that by hand? Me neither. If you need n back to its usual self for some reason, always remember you can reset a variable in Maple using the command

```
n:='n';
```