1a/ $A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & 0 & -2 \\ 0 & \lambda & 0 \\ -2 & 0 & \lambda \end{pmatrix}$$

$$= \lambda \det\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - 0 \cdot \det\begin{pmatrix} 0 & 0 \\ -2 & \lambda \end{pmatrix} + (-2)\det\begin{pmatrix} 0 & \lambda \\ -2 & 0 \end{pmatrix}$$

$$= \lambda^3 - 4\lambda = \lambda(\lambda-2)(\lambda+2)$$

<u>EIGENVALUES:</u> $\lambda_1 = -2$
$\lambda_2 = 0$
$\lambda_3 = 2$

TO FIND EIGENVECTORS, SOLVE $(\lambda I - A)v = 0$ for $v \neq 0$.

λ_1: $(\lambda_1 I - A)v_1 = 0 \Rightarrow \begin{pmatrix} -2 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \alpha = -\gamma \\ \beta = 0 \end{matrix}$

$\Rightarrow v_1 = \begin{pmatrix} \alpha \\ 0 \\ -\alpha \end{pmatrix}$ or, normalized: $\underline{v_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$

λ_2: $(\lambda_2 I - A)v_2 = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \alpha = \gamma = 0$

$\Rightarrow v_2 = \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}$; normalized: $\underline{v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

λ_3: $(\lambda_3 I - A)v_3 = 0 \Rightarrow \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \beta = 0 \\ \alpha = \gamma \end{matrix}$

$\Rightarrow v_3 = \begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix}$ or, normalized: $\underline{v_3 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$.

1b/ $A_\alpha = A + \alpha I$. Suppose $Av = \lambda v$. Then $(v \neq 0)$

$$A_\alpha v = (A + \alpha I)v = Av + \alpha v = \lambda v + \alpha v = (\lambda + \alpha)v$$

Hence v is an eigenvector of A_α with eigenvalue $\hat{\lambda} = \lambda + \alpha$.

1c/ Solve $A_\alpha x_\alpha = b$, where $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Write $b = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$= 0\, v_1 + 1 \cdot v_2 + \sqrt{2}\, v_3 \quad \text{By inspection.}$$

Spectral method: $x_\alpha = \sum_{k=1}^{3} \frac{c_k}{\lambda_k + \alpha} v_k$

since $\{\lambda_k + \alpha\}$ are the eigenvalues of A_α by question 1b. We compute:

$$x_\alpha = \frac{0}{\lambda_1 + \alpha} v_1 + \frac{1}{\lambda_2 + \alpha} v_2 + \frac{\sqrt{2}}{\lambda_3 + \alpha} v_3$$

$$= \frac{1}{\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{\sqrt{2}}{2+\alpha} \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/(2+\alpha) \\ 1/\alpha \\ 1/(2+\alpha) \end{pmatrix}}}$$

This procedure fails when $\alpha = 0$ and $\alpha = -2$.

In this case A is singular and <u>no solution</u> exists.

If $\alpha = 2$ then we would have $\frac{0}{\lambda_1 + \alpha} = \frac{0}{0}$ and we need to be careful. A is singular, but there are <u>infinitely many solutions</u> of the form

$$x_\alpha = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix} \quad \text{for any } \delta.$$

2a/ $Lu = -\dfrac{d^2u}{dx^2}$, $u(0) = u(\pi) = 0$

Suppose $Lu = \lambda u. \Rightarrow -\dfrac{d^2u}{dx^2} = \lambda u.$

$\Rightarrow u(x) = A\sin(\sqrt{\lambda}\,x) + B\cos(\sqrt{\lambda}\,x)$

$u(0) = 0 \Rightarrow \quad 0 = A\sin(0) + B\cos(0) \Rightarrow \underline{B = 0}$

$u(\pi) = 0 \Rightarrow \quad 0 = A\sin(\sqrt{\lambda}\,\pi)$.

Pick λ SO THAT $\sin(\sqrt{\lambda}\,\pi) = 0$.

Thus $\sqrt{\lambda} = n$ for $n = 1, 2, \ldots$, so $\lambda = n^2$.

EIGENVALUES: $\lambda_n = n^2$
EIGENFUNCTIONS: $u_n(x) = \sin(nx)$ $\Big\}$ $n = 1, 2, 3, \ldots$

2b/ Using 1(b), we guess:

$$\lambda_n = n^2 + \alpha, \quad u_n(x) = \sin(nx)$$

(EIGENVALUES ARE SHIFTED, EIGENFUNCTIONS STAY THE SAME)

A simple calculation verifies this guess:

$$L_\alpha u_n = -\frac{d^2u_n}{dx^2} + \alpha u_n = n^2\sin(nx) + \alpha\sin(nx)$$
$$= (n^2 + \alpha)\sin(nx) = \lambda_n u_n(x).$$

2c/ Two steps. WRITE: $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(f, u_n)}{(u_n, u_n)} u_n(x)$

SOLVE: $u(x) = \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^2+\alpha}\right) \frac{(f, u_n)}{(u_n, u_n)} u_n(x)$.

2d/ Use 2b,c with $\alpha = 1$.

$$(f, u_n) = (\sin(7x), \sin(nx)) = \begin{cases} 0 & n \neq 7 \\ (u_n, u_n) & n = 7 \end{cases}$$

by orthogonality of eigenfunctions.

Thus $$u(x) = \sum_{n=1}^{\infty} \frac{1}{n^2+1} \frac{(f, u_n)}{(u_n, u_n)} u_n(x)$$
$$= \frac{1}{7^2+1} \frac{(u_7, u_7)}{(u_7, u_7)} u_7(x) = \frac{1}{50}\sin(7x)$$

(NO INTEGRATION NEEDED!)

3a/ Form the GRAM matrix equation. $V_N = \sum_{j=1}^{N} c_j \phi_j$, where

$$\begin{pmatrix} (\phi_1,\phi_1) & \cdots & (\phi_1,\phi_N) \\ \vdots & \ddots & \vdots \\ (\phi_N,\phi_1) & \cdots & (\phi_N,\phi_N) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} (f,\phi_1) \\ \vdots \\ (f,\phi_N) \end{pmatrix}$$

Solve this system for $c_1, \dots, c_N$.

3b/ No. Different bases result in different coefficients, but V_N must remain the same:

$$\|v - v_N\| = \min_{\hat{v} \in V_N} \|v - \hat{v}\| \quad \text{is independent of the basis.}$$

3c/ Yes. The inner product defines the norm: $\|x\| = (x,x)^{1/2}$, so different inner products give different measures of distance. Best approximation in the energy inner product is different from best approximation in the standard inner product.

3d/ $(\phi_1,\phi_1) = \int_0^1 1\cdot 1\, dx + \int_0^1 0\cdot 0\, dx = \underline{1}$

$(\phi_1,\phi_2) = (\phi_2,\phi_1) = \int_0^1 1\cdot(1-2x)\,dx + \int_0^1 0\cdot(-2)\,dx = \int_0^1 (1-2x)\,dx = \underline{0}$

$(\phi_2,\phi_2) = \int_0^1 (1-2x)^2\,dx + \int_0^1 (-2)^2\,dx = \int_0^1 (1-4x+4x^2)+4\,dx$
$= 13/3.$

$(f,\phi_1) = \int_0^1 e^x\,dx = e^1 - e^0 = \underline{e-1}$

$(f,\phi_2) = \int_0^1 e^x(1-2x)\,dx + \int_0^1 -2\cdot e^x\,dx = e^x\big|_0^1 - 2\int_0^1 xe^x\,dx - 2e^x\big|_0^1$
$= \underline{-1-e.}$

$[\int_0^1 xe^x\,dx = 1]$ (integration by parts)

Solve: $\begin{pmatrix} 1 & 0 \\ 0 & 13/3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e-1 \\ -1-e \end{pmatrix} \Rightarrow \begin{array}{l} c_1 = e-1 \\ c_2 = \frac{3}{13}(-1-e) \end{array}$

Best approximation:

$$v_2(x) = \underline{(e-1)\cdot 1 + \tfrac{3}{13}(-1-e)(1-2x)} = \underline{\tfrac{2}{13}(5e-8) + \tfrac{6x}{13}(e+1)}$$
$$= c_1\phi_1 + c_2\phi_2$$

4a/ $Lu = \dfrac{d^2}{dx^2}\left(k(x)\dfrac{d^2u}{dx^2}(x)\right) \qquad u(0)=u(1)=\dfrac{du}{dx}(0)=\dfrac{du}{dx}(1)=0.$

Suppose $u,v \in C_D^4[0,1]$. $\Rightarrow \begin{cases} v(0)=v(1)=\frac{dv}{dx}(0)=\frac{dv}{dx}(1)=0 \\ u(0)=u(1)=\frac{du}{dx}(0)=\frac{du}{dx}(1)=0. \end{cases}$

$$(Lu,v) = \int_0^1 \frac{d^2}{dx^2}\left(k(x)\frac{d^2u}{dx^2}(x)\right)v(x)\,dx$$

I.B.P.
$$= \underbrace{\left[\frac{d}{dx}\left(k(x)\frac{d^2u}{dx^2}(x)\right)v(x)\right]_0^1}_{=0 \text{ since } v(0)=v(1)=0} - \int_0^1 \frac{d}{dx}\left(k(x)\frac{d^2u}{dx^2}(x)\right)\frac{dv}{dx}(x)\,dx$$

(*)
$$= -\underbrace{\left[k(x)\frac{d^2u}{dx^2}(x)\frac{dv}{dx}(x)\right]_0^1}_{=0 \text{ since } \frac{dv}{dx}(0)=\frac{dv}{dx}(1)=0} + \int_0^1 k(x)\frac{d^2u}{dx^2}(x)\frac{d^2v}{dx^2}(x)\,dx$$

$$= \underbrace{\left[\frac{du}{dx}(x)\left(k(x)\frac{d^2v}{dx^2}(x)\right)\right]_0^1}_{=0 \text{ since } \frac{du}{dx}(0)=\frac{du}{dx}(1)=0} - \int_0^1 \frac{du(x)}{dx}\left(\frac{d}{dx}\left(k(x)\frac{d^2v}{dx^2}(x)\right)\right)dx$$

$$= \underbrace{\left[u(x)\frac{d}{dx}\left(k(x)\frac{d^2v}{dx^2}(x)\right)\right]_0^1}_{=0 \text{ since } u(0)=u(1)=0} + \int_0^1 u(x)\left(\frac{d^2}{dx^2}\left(k(x)\frac{d^2v(x)}{dx^2}\right)\right)dx$$

$= (u, Lv)$ Hence L is symmetric.

4b/ To derive the weak form, multiply $Lu=f$ by a test function $v \in C_D^4[0,1]$ and integrate:

$Lu=f \Rightarrow (Lu,v)=(f,v) \quad \forall\, v \in C_D^4[0,1].$

$$(f,v) = (Lu,v) = \int_0^1 \frac{d^2}{dx^2}\left(k(x)\frac{d^2u}{dx}(x)\right)v(x)\,dx$$

$$= \cancel{\left[\frac{d}{dx}\left(k(x)\frac{d^2u}{dx^2}(x)\right)v(x)\right]_0^1}^{\,0} - \int_0^1 \frac{d}{dx}\left(k(x)\frac{d^2u}{dx^2}(x)\right)\frac{dv}{dx}(x)\,dx$$

$$= -\cancel{\left[k(x)\frac{d^2u}{dx^2}(x)\frac{dv}{dx}(x)\right]_0^1}^{\,0} + \int_0^1 k(x)\frac{d^2u}{dx^2}(x)\frac{d^2v}{dx^2}(x)\,dx$$

$$= \int_0^1 k(x)\frac{d^2u}{dx^2}(x)\frac{d^2v}{dx^2}(x)\,dx$$

4b, continued

Weak Form: Find $u \in C_D^4[0,1]$ such that

$$a(u,v) = (f,v) \quad \text{for all } v \in C_D^4[0,1]$$

where $a(u,v) = \int_0^1 k(x) \frac{d^2u}{dx^2}(x) \frac{d^2v}{dx^2}(x)\, dx$.

4c

WE MUST SHOW THAT $\underline{a(\cdot,\cdot) \text{ IS AN INNER PRODUCT}}$:

- BY INSPECTION, $a(u,v) = a(v,u)$.
- BY LINEARITY OF INTEGRATION & DIFFERENTIATION,
  $$a(\alpha u + \beta v, w) = \alpha\, a(u,w) + \beta\, a(v,w).$$
- POSITIVITY REQUIRES MORE CARE:
  $$a(u,u) = \int_0^1 k(x) \left(\frac{d^2u}{dx^2}(x)\right)^2 dx \geq 0 \quad \text{since } k(x) \geq 0.$$

WHEN CAN $a(u,u) = 0$? Only when $\frac{d^2u}{dx^2}(x) = 0$ for all $x \in [0,1]$.

THIS IMPLIES THAT $u(x) = A + Bx$ for some $A, B \in \mathbb{R}$.

BUT $u(0) = 0 \Rightarrow A = 0$ and $\frac{du}{dx}(0) = 0 \Rightarrow B = 0$.

HENCE $a(u,u) = 0$ only when $u(x) = 0$ for all $x \in [0,1]$.

<u>**GALERKIN PROBLEM:**</u> Find $u_N \in V_N$ such that

$$a(u_N, v) = (f,v) \quad \text{for all } v \in V_N.$$

EQUIVALENTLY, $a(u_N, \phi_j) = (f, \phi_j) \quad j = 1, \ldots, N.$

WRITE $u_N = \sum_{k=1}^N c_k \phi_k \Longrightarrow \sum_{k=1}^N c_k (\phi_k, \phi_j) = (f, \phi_j) \quad j = 1, \ldots, N.$

THUS

$$\begin{pmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_1,\phi_N) \\ \vdots & & \vdots \\ a(\phi_N,\phi_1) & & a(\phi_N,\phi_N) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} (f,\phi_1) \\ \vdots \\ (f,\phi_N) \end{pmatrix}$$

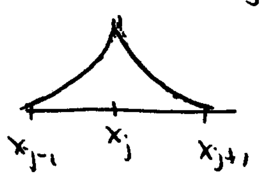
$$K\underline{u} = \underline{f}.$$

4d/ HAT FUNCTIONS ARE ENTIRELY UNSUITABLE FOR THIS PROBLEM; SINCE THEY ARE PIECEWISE LINEAR, WE HAVE $\underline{a(\phi_j, \phi_k) = 0}$ FOR ALL j, k. Thus, $K = 0$ in this case.

WE NEED AN ALTERNATIVE SET OF BASIS FUNCTIONS COMPRISED, FOR EXAMPLE, OF PIECEWISE <u>QUADRATIC</u> POLYNOMIALS.

SOME STUDENTS PROPOSED SOMETHING LIKE

$$\phi_j(x) = \begin{cases} 0 & x \notin [x_{j-1}, x_{j+1}] \\ \dfrac{(x - x_{j-1})^2}{h^2} & x \in [x_{j-1}, x_j] \\ \dfrac{(x - x_{j+1})^2}{h^2} & x \in [x_j, x_{j+1}] \end{cases}$$



which would give a matrix K that was <u>tridiagonal</u> and would additionally satisfy $\frac{du_N}{dt}(0) = \frac{du_N}{dt}(1) = 0$, which hat functions wouldn't satisfy. For greater continuity (of derivatives), one could use <u>cubic</u> <u>splines</u>, piecewise cubic polynomials supported on <u>four</u> intervals $[x_{j-2}, x_{j+2}]$, leading to a matrix with <u>seven</u> nonzero entries per row, and an approximation U_N that satisfies all four boundary conditions and is additionally continuous with two continuous derivatives on $[0,1]$: $U_N \in C^2[0,1]$.