

# CAAM 336 – Differential Equations Midterm Exam

October 14, 2005

## Instructions:

1. This exam is due **at the start of class (1 pm) on Monday, October 24, 2005**. Do not place your exams in the instructor's mailbox. Exams will not be accepted after the due date except under extraordinary circumstances, with sufficient advance notice, and with appropriate documentation.
2. You have **five (5) contiguous hours** to complete this exam. You must record your start and end dates/times on the lines below.
3. You must work **completely independently**: no notes, books, former homeworks or exams, calculators, computers, cell phones, friends, strangers, etc. Just you and the test. Any questions should be directed to the instructor or one of the teaching assistants (Tommy Binford and Josef Sifuentes).
4. You must provide **clear, convincing, and complete** arguments for all of your answers; credit will be given only for what you actually write, not for what you know but did not write. Sloppy or disorganized work or equations without accompanying written explanations will not receive full credit. Your work should be presented **neatly** on scratch paper, which should be stapled together with the exam itself.
5. You must write out and sign the **pledge** in the space below.

**Start Date/Time:**

**End Date/Time:**

**Pledge:**

1. (15 points) Consider the following differential equations for the unknown function  $u$ :

(a)  $\frac{d}{dx} \left( \varepsilon(x) \frac{du}{dx} \right) + c^2 u = 0$  (Wave Propagation in a Dielectric)

(b)  $\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 - 1 = 0$  (Geometrical Optics)

(c)  $\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx \frac{\partial v}{\partial x} = rv(x, t)$  (Value of a Stock Option)

(d)  $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$  (Traffic Flow)

Answer each of the following questions. Where appropriate, you can refer to the equations above by their letter labels: a, b, c, and d. For this problem only, no justification of your answers is required.

- i. What is the order of each equation?
- ii. Which equations are ordinary differential equations?
- iii. Which equations are non-linear?
- iv. Of the linear equations, which are homogeneous?
- v. Of the linear equations, which ones have constant coefficients?

2. (20 points) *Spectral Theorem/Spectral Method.*

(a) State the Spectral Theorem for real, symmetric,  $n \times n$  matrices. In particular, what do we know about the eigenvalues and eigenvectors of such matrices?

(b) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solve  $\mathbf{Ax} = \mathbf{b}$  using the “spectral method”. Show all of your work.

(c) Using the spectral method for linear, symmetric differential operators (i.e., the Fourier series method), solve the boundary value problem:

$$-\frac{d^2 u}{dx^2} = \cos\left(\frac{\pi x}{2}\right) + \cos\left(\frac{9\pi x}{2}\right), \quad 0 < x < 1,$$
$$\frac{du}{dx}(0) = 0, \quad u(1) = 0.$$

You must derive the eigenvalues and eigenvectors of the differential operator. Show all of your work. (*Hint:* The solution has a relatively simple form.)

3. (25 points) *Best Approximation.*

(a) State the Best Approximation Theorem.

(b) Find the line that gives the “best approximation” to the following three points:

$$(x_1, y_1) = (0, 1)$$

$$(x_2, y_2) = (1, 0)$$

$$(x_3, y_3) = (2, 0)$$

Plot (by hand) these points and the line that you find on the same graph. Compute the error in the approximation. In what sense is this line the “best approximation” to the points?

- (c) Consider the boundary value problem:

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 < x < \ell$$

$$u(0) = 0, \quad u(\ell) = 0.$$

In what sense is the Fourier series (sine series) solution to this equation the “best approximation”? (*Hint*: Your answer should refer to the space  $F_N = \{\sin(\frac{\pi x}{\ell}), \dots, \sin(\frac{N\pi x}{\ell})\}$ .)

- (d) Given an energy inner product  $a(u, v)$  defined for all  $u, v \in V$ , let  $u$  solve the weak form

$$a(u, v) = (f, v) \quad \text{for all } v \in V$$

and let  $v_n$  solve the weak form

$$a(v_n, v) = (f, v) \quad \text{for all } v \in V_n$$

where  $V_n$  is a finite-dimensional subspace of  $V$ . Show that  $v_n$  is the best approximation to  $u$  from  $V_n$ , and explain precisely what this means.

4. (20 points) *Finite Elements*. Let the energy inner product be defined as

$$a(u, v) = \int_0^\ell \left[ \kappa(x) \frac{du}{dx} \frac{dv}{dx} + p(x)u(x)v(x) \right] dx,$$

for all  $u, v \in V = C_D^2[0, \ell]$ , where  $\kappa(x) > 0$  and  $p(x) > 0$  for all  $0 < x < \ell$ . Assume that the weak form of a differential equation is the following: find  $u \in V$  such that

$$a(u, v) = (f, v), \quad \text{for all } v \in V.$$

Let  $S_n$  be the space of continuous, piecewise-linear functions that are associated with the regular mesh with spacing  $\ell/n$  and that are equal to zero at  $x = 0$  and  $x = \ell$ , and let  $\phi_1, \dots, \phi_{n-1}$  be the “hat” basis functions.

- (a) Derive the finite element formulation of the problem, i.e., reduce the weak form above to a finite-dimensional approximation to the problem, and express this finite-dimensional problem as a matrix equation  $\mathbf{K}\mathbf{u} = \mathbf{f}$ , with appropriate definitions of  $\mathbf{K}$ ,  $\mathbf{u}$ , and  $\mathbf{f}$ . Show all of your work. (*Note*: You can leave everything in terms of integrals involving the basis functions; you do not need to compute the integrals.)
- (b) Calculate  $K_{12}$  and  $K_{13}$  in terms of  $\ell$  and  $n$  (assuming  $n > 3$ ) given that  $k = 1$  and  $p = 2$ .
5. (20 points) *Fourier Series*. Given a thin bar of length  $\ell = \pi$  cm with  $\rho = 1$  g/cm<sup>3</sup>,  $c = 1$  J/gK, and  $\kappa = 1$  W/cmK, assume that the bar is initially at a uniform temperature of 0°C. Then, beginning at time zero, heat is added at a rate of  $f(x, t) = \sin(x)$  while the ends of the bar are held constant at 0°C.

- (a) Set up the corresponding initial-boundary-value problem.
- (b) Solve this problem using Fourier series methods. Determine the steady-state solution by taking a limit. Draw a rough sketch of how the solution will evolve with time. (*Note*: you must show how to derive the solution from first principles and may not make use of a memorized formula.) (*Hint*: The solution has a relatively simple form.)