

# CAAM 336 – Differential Equations Midterm Exam Fall 2005 SOLUTIONS

1. (15 points) Consider the following differential equations for the unknown function  $u$ :

- (a)  $\frac{d}{dx} \left( \varepsilon(x) \frac{du}{dx} \right) + c^2 u = 0$  (Wave Propagation in a Dielectric)
- (b)  $\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 - 1 = 0$  (Geometrical Optics)
- (c)  $\frac{\partial v}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + rx \frac{\partial v}{\partial x} = rv(x, t)$  (Value of a Stock Option)
- (d)  $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$  (Traffic Flow)

Answer each of the following questions. Where appropriate, you can refer to the equations above by their letter labels: a, b, c, and d. For this problem only, no justification of your answers is required.

- i. What is the order of each equation?
- ii. Which equations are ordinary differential equations?
- iii. Which equations are non-linear?
- iv. Of the linear equations, which are homogeneous?
- v. Of the linear equations, which ones have constant coefficients?

**Solution:**

**Order of equations?** (b) and (d) are first order, (a) and (c) are second order

**Which are ODEs?** only (a)

**Which are non-linear?** (b), because of squared derivatives, and (d) because of  $u \frac{\partial u}{\partial x}$

**Of the linear equations, which are homogeneous?** (a) and (c)

**Of the linear equations, which ones have constant coefficients?** None of them

2. (20 points) *Spectral Theorem/Spectral Method.*

- (a) State the Spectral Theorem for real, symmetric,  $n \times n$  matrices. In particular, what do we know about the eigenvalues and eigenvectors of such matrices?
- (b) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solve  $\mathbf{Ax} = \mathbf{b}$  using the “spectral method”. Show all of your work.

- (c) Using the spectral method for linear, symmetric differential operators (i.e., the Fourier series method), solve the boundary value problem:

$$-\frac{d^2 u}{dx^2} = \cos\left(\frac{\pi x}{2}\right) + \cos\left(\frac{9\pi x}{2}\right), \quad 0 < x < 1,$$

$$\frac{du}{dx}(0) = 0, \quad u(1) = 0.$$

You must derive the eigenvalues and eigenvectors of the differential operator. Show all of your work. (*Hint*: The solution has a relatively simple form.)

**Solution:**

(a) If  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is symmetric, then we have:

- Every eigenvalue of  $\mathbf{A}$  is real (and the corresponding eigenvectors can be chosen to be real).
- If  $\mathbf{x}_1, \mathbf{x}_2$  are eigenvectors corresponding to distinct eigenvalues, then  $\mathbf{x}_1, \mathbf{x}_2$  are orthogonal
- There exists an orthogonal basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $\mathbf{A}$ .

(b) We first determine and solve the characteristic polynomial:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -2 \\ 0 & -2 & \lambda \end{bmatrix} = \lambda^2(\lambda - 1) - 4(\lambda - 1) = (\lambda - 1)(\lambda^2 - 4) = (\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$$

We see our eigenvalues are  $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 1$ . We now compute the first eigenvector,  $\mathbf{u}_1$ :

$$(\lambda_1 \mathbf{I} - \mathbf{A})\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{u}_1 = 0$$

We see the reduced form of our matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ , so we need to solve the equations:

$$\begin{aligned} \mathbf{u}_1(1) &= 0 \\ \mathbf{u}_1(2) - \mathbf{u}_1(3) &= 0 \end{aligned}$$

And by inspection, we realize we can set  $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ , as we notice the last 2 entries of this eigenvector need to be the same, and we can just put a zero in the first entry. Now on to  $\mathbf{u}_2$ :

$$(\lambda_2 \mathbf{I} - \mathbf{A})\mathbf{u}_2 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} \mathbf{u}_2 = 0$$

We see the reduced form of our matrix is  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , so we need to solve the equations:

$$\begin{aligned}\mathbf{u}_2(1) &= 0 \\ \mathbf{u}_2(2) + \mathbf{u}_2(3) &= 0\end{aligned}$$

And by inspection, we realize we can set  $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ , as we can set the first entry to 0, and the next two need to be equal but opposite in sign. Now for  $\mathbf{u}_3$ :

$$(\lambda_3 \mathbf{I} - \mathbf{A})\mathbf{u}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix} \mathbf{u}_3 = 0$$

We see the reduced form of our matrix is  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ , so we need to solve the equations:

$$\begin{aligned}\mathbf{u}_3(2) &= 0 \\ \mathbf{u}_3(3) &= 0\end{aligned}$$

And by inspection, we set  $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ , as we see the first entry can be anything and the others need to be zero. Because  $\mathbf{A}$  is symmetric and real, we know eigenvectors are orthogonal (very easy to check this). We now employ the spectral method and determine  $\mathbf{x}$ :

$$\begin{aligned}\mathbf{x} &= \frac{1}{\lambda_1} \frac{(\mathbf{u}_1 \cdot \mathbf{b})}{(\mathbf{u}_1 \cdot \mathbf{u}_1)} \mathbf{u}_1 + \frac{1}{\lambda_2} \frac{(\mathbf{u}_2 \cdot \mathbf{b})}{(\mathbf{u}_2 \cdot \mathbf{u}_2)} \mathbf{u}_2 + \frac{1}{\lambda_3} \frac{(\mathbf{u}_3 \cdot \mathbf{b})}{(\mathbf{u}_3 \cdot \mathbf{u}_3)} \mathbf{u}_3 \\ &= \frac{1}{2} \cdot \frac{5}{4} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{-2} \cdot \frac{-1}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{1} \cdot \frac{1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix}.\end{aligned}$$

(c): We can prove, using integration by parts, that our second derivative operator is symmetric, so it has real eigenvalues: for every  $u, v \in C_m^2[0, 1] = \{u \in C^2[0, 1] : \frac{du}{dx}(0) = u(1) = 0\}$ ,

$$(L_m u, v) = - \int_0^1 \frac{d^2 u}{dx^2} v(x) dx = \int_0^1 \frac{du}{dx} \frac{dv}{dx} dx = - \int_0^1 u(x) \frac{d^2 v}{dx^2} dx = (u, L_m v).$$

Therefore,  $L_m$  is a symmetric linear operator.

We can again use integration by parts to prove that its eigenvalues are actually positive. Let  $(\lambda, u)$  be an eigenpair:

$$\lambda(u, u) = (\lambda u, u) = (L_m u, u) = - \int_0^1 \frac{d^2 u}{dx^2} u(x) dx = \int_0^1 \left( \frac{du}{dx} \right)^2 dx \geq 0.$$

In fact, this expression is actually strictly positive since it could be zero if and only if  $\frac{du}{dx} = 0$  for all  $x \in [0, 1]$ , which would imply that  $u$  is a constant. However, since  $u(1) = 0$ ,  $u$  would

have to be identically zero, which means it could not be an eigenvalue, contradicting our original assumption. Therefore, all eigenvalues must be positive.

Now to solve the differential equation, we first find the eigenpairs. Since  $\lambda > 0$ , we define  $\lambda = \theta^2$ , where we may assume  $\theta > 0$ , and solve:

$$\begin{aligned}\frac{d^2u}{dx^2} + \theta^2u &= 0 \\ \frac{du}{dx}(0) &= 0 \\ u(1) &= 0\end{aligned}$$

The general solution is  $u(x) = c_1 \cos(\theta x) + c_2 \sin(\theta x)$ . We differentiate to get  $\frac{du}{dx} = -\theta c_1 \sin(\theta x) + \theta c_2 \cos(\theta x)$ . Using the boundary condition  $\frac{du}{dx}(0) = 0$ , we see that

$$0 = \theta c_2$$

and since  $\theta > 0$ ,  $c_2 = 0$ . Thus any nonzero eigenfunction must be a multiple of  $\cos(\theta x)$ . Looking at our other boundary condition,  $u(1) = 0$ , we see that:

$$c_1 \cos(\theta) = 0$$

thus we have that  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, \frac{(2n-1)\pi}{2}, \dots$ . Thus our eigenvalues are

$$\lambda_n = \frac{(2n-1)^2\pi^2}{4}$$

for  $n = 1, 2, \dots$ . Our eigenfunctions are:

$$\phi_n(x) = \cos\left(\frac{(2n-1)\pi}{2}x\right)$$

The eigenfunctions are orthogonal, and can be verified directly if so desired. We now write  $f$  out fully as a Fourier series:

$$f(x) = \sum_{n=1}^{\infty} \frac{(\phi_n, f)}{(\phi_n, \phi_n)} \phi_n$$

We see that

$$(\phi_n, \phi_n) = \int_0^1 \cos^2\left(\frac{(2n-1)\pi}{2}x\right) dx = \frac{1}{2}$$

and

$$\begin{aligned}(\phi_n, f) &= \int_0^1 f(x) \cos\left(\frac{(2n-1)\pi}{2}x\right) dx \\ &= \int_0^1 \cos\left(\frac{\pi x}{2}\right) \cos\left(\frac{(2n-1)\pi}{2}x\right) dx + \int_0^1 \cos\left(\frac{9\pi x}{2}\right) \cos\left(\frac{(2n-1)\pi}{2}x\right) dx\end{aligned}$$

But now we notice that  $f$  consists of basis functions as well! Again from orthogonality of our basis functions, both of these integrals are going to be zero unless  $\frac{\pi}{2} = \frac{(2n-1)\pi}{2}$ , or  $\frac{9\pi}{2} = \frac{(2n-1)\pi}{2}$ . We find that the only terms that fall through are for  $n = 1, 5$ . And for

these particular values of  $n$ , we also see that the other integral drops out, meaning that  $(\phi_1, f) = (\phi_1, \phi_1)$  and  $(\phi_5, f) = (\phi_5, \phi_5)$ . Thus our RHS is:

$$f(x) = \frac{(\phi_1, f)}{(\phi_1, \phi_1)}\phi_1 + \frac{(\phi_5, f)}{(\phi_5, \phi_5)}\phi_5 = \phi_1 + \phi_5$$

Now we assume our solution  $u$  is written as a Fourier series,  $u(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$ , and we want to solve  $\mathbf{L}_m u = f$ , giving us

$$\sum_{n=1}^{\infty} \frac{(2n-1)^2 \pi^2}{4} a_n \phi_n = f = \phi_1 + \phi_5$$

and again due to the orthogonality of our basis functions, only the first and fifth term fall through, all other Fourier coefficients are zero. Equating the coefficients and solving for  $a_1, a_5$ , we see

$$a_1 = \frac{4}{\pi^2}$$

$$a_5 = \frac{4}{9^2 \pi^2}$$

and so our solution is

$$u(x) = a_1 \phi_1(x) + a_5 \phi_5(x) = \frac{4}{\pi^2} \cos\left(\frac{\pi x}{2}\right) + \frac{4}{9^2 \pi^2} \cos\left(\frac{9\pi x}{2}\right)$$

3. (25 points) *Best Approximation.*

- (a) State the Best Approximation Theorem.  
 (b) Find the line that gives the “best approximation” to the following three points:

$$(x_1, y_1) = (0, 1)$$

$$(x_2, y_2) = (1, 0)$$

$$(x_3, y_3) = (2, 0)$$

Plot (by hand) these points and the line that you find on the same graph. Compute the error in the approximation. In what sense is this line the “best approximation” to the points?

- (c) Consider the boundary value problem:

$$-\frac{d^2 u}{dx^2} = f(x), \quad 0 < x < \ell$$

$$u(0) = 0, \quad u(\ell) = 0.$$

In what sense is the Fourier series (sine series) solution to this equation the “best approximation”? (*Hint:* Your answer should refer to the space  $F_N = \text{span} \left\{ \sin\left(\frac{\pi x}{\ell}\right), \dots, \sin\left(\frac{N\pi x}{\ell}\right) \right\}$ .)

- (d) Given an energy inner product  $a(u, v)$  defined for all  $u, v \in V$ , let  $u$  solve the weak form

$$a(u, v) = (f, v) \quad \text{for all } v \in V$$

and let  $v_n$  solve the weak form

$$a(v_n, v) = (f, v) \quad \text{for all } v \in V_n$$

where  $V_n$  is a finite-dimensional subspace of  $V$ . Show that  $v_n$  is the best approximation to  $u$  from  $V_n$ , and explain precisely what this means.

**Solution:**

(a) Let  $V$  be a vector space with inner product  $(\cdot, \cdot)$ , let  $W$  be a finite-dimensional subspace of  $V$  and let  $\mathbf{v} \in V$ . Then we have:

- There is a unique  $\mathbf{u} \in W$  such that

$$\|\mathbf{v} - \mathbf{u}\| = \min_{\mathbf{w} \in W} \|\mathbf{v} - \mathbf{w}\|$$

Meaning there is a unique best approximation to  $\mathbf{v}$  from  $W$ . We also call  $\mathbf{u}$  the projection of  $\mathbf{v}$  onto  $W$ , and write  $\mathbf{u} = \text{proj}_W \mathbf{v}$ .

- A vector  $\mathbf{u} \in W$  is the best approximation to  $\mathbf{v}$  from  $W$  if and only if  $(\mathbf{v} - \mathbf{u}, \mathbf{z}) = 0$  for all  $\mathbf{z} \in W$ .

(b) We ideally would like to solve the following linear system:

$$\begin{aligned} ax_1 + b &= y_1 & b &= 1 \\ ax_2 + b &= y_2 & \Rightarrow a + b &= 0 \\ ax_3 + b &= y_3 & 2a + b &= 0 \end{aligned}$$

But we can see we cannot find such  $a$  and  $b$ , so we solve it using our best approximation

theorem. We let  $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ ,  $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ , and we formulate the Gram matrix:

$$\mathbf{G} = \begin{bmatrix} \mathbf{x} \cdot \mathbf{x} & \mathbf{x} \cdot \mathbf{e} \\ \mathbf{x} \cdot \mathbf{e} & \mathbf{e} \cdot \mathbf{e} \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}$$

and our right hand side becomes

$$\begin{bmatrix} \mathbf{x} \cdot \mathbf{y} \\ \mathbf{e} \cdot \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then solving the linear system  $\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  we get  $a = -\frac{1}{2}$ ,  $b = \frac{5}{6}$ .

We now compute the error:

$$\text{Error} = \|(a\mathbf{x} + b) - \mathbf{y}\| = \sqrt{\left(-\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{6}\right)^2} = \frac{1}{\sqrt{6}}$$

The line is the best approximation in the sense that for all possible lines, it minimizes the 2 norm error, which minimizes the sum of the squared vertical distances from the line to the given points.

(c) The attention here is drawn to the way in which we say “best approximation”. When calculating the Fourier series solution in this class, we use the  $L^2$  inner product, as  $(u, v) = \int_a^b u(x)v(x)dx$ . Thus our Fourier series solution minimizes the induced  $L^2$  norm,  $\|u\| = \sqrt{(u, u)} = \sqrt{\int_a^b u(x)^2 dx}$  of the error in our solution, meaning if  $u$  is the exact solution and  $u^F$  is our Fourier series solution, then  $\sqrt{\int_a^b (u(x) - u^F(x))^2 dx}$  is the minimum. More precisely,

$u^F$  is the function from  $F_N$  that minimizes the norm of the error, i.e.,  $\|u - u^F\|$  is smaller than  $\|u - v\|$  for any function  $v$  of the form  $\sum_{n=1}^N c_n \sin\left(\frac{n\pi x}{L}\right)$ . In other words, the best choice for the constants  $c_n$  are the Fourier coefficients.

(d) We know from the projection theorem that a  $v$  is the best approximation to  $u$  from a finite dimensional subspace  $W$  if and only iff  $(u - v, z) = 0$  for all  $z \in W$ . Here our inner product is the energy inner product,  $a(\cdot, \cdot)$ . For all  $v \in V_n$ , we know  $a(v_n, v) = (f, v)$ . Also,  $a(u, v) = (f, v)$ , since this holds for the entire space  $V$  and  $V_n$  is a subspace of  $V$  (i.e.  $V_n \subset V$ ). So  $a(u - v_n, v) = (f, v) - (f, v) = 0$ , showing that  $v_n$  is the best approximation to  $u$  from  $V_n$ .

Here, best approximation means that we minimize the error in our norm, which is the energy norm induced by our energy inner product, so we minimize  $\|u - v\|_E$  over all functions  $v \in V_n$ . Thus,  $v_n$  is the function from the set  $V_n$  that minimizes this norm among all possible elements in  $V_n$ .

4. (20 points) *Finite Elements*. Let the energy inner product be defined as

$$a(u, v) = \int_0^\ell \left[ \kappa(x) \frac{du}{dx} \frac{dv}{dx} + p(x)u(x)v(x) \right] dx,$$

for all  $u, v \in V = C_D^2[0, \ell]$ , where  $\kappa(x) > 0$  and  $p(x) > 0$  for all  $0 < x < \ell$ . Assume that the weak form of a differential equation is the following: find  $u \in V$  such that

$$a(u, v) = (f, v), \quad \text{for all } v \in V.$$

Let  $S_n$  be the space of continuous, piecewise-linear functions that are associated with the regular mesh with spacing  $\ell/n$  and that are equal to zero at  $x = 0$  and  $x = \ell$ , and let  $\phi_1, \dots, \phi_{n-1}$  be the “hat” basis functions.

- (a) Derive the finite element formulation of the problem, i.e., reduce the weak form above to a finite-dimensional approximation to the problem, and express this finite-dimensional problem as a matrix equation  $\mathbf{K}\mathbf{u} = \mathbf{f}$ , with appropriate definitions of  $\mathbf{K}$ ,  $\mathbf{u}$ , and  $\mathbf{f}$ . Show all of your work. (*Note*: You can leave everything in terms of integrals involving the basis functions; you do not need to compute the integrals.)
- (b) Calculate  $K_{12}$  and  $K_{13}$  in terms of  $\ell$  and  $n$  (assuming  $n > 3$ ) given that  $k = 1$  and  $p = 2$ .

**Solution:**

(a) We first determine the finite dimensional form of the problem, which is to determine  $v_n$  such that

$$a(v_n, v) = (f, v) \quad \forall v \in S_n.$$

But we know to satisfy this  $\forall v$ , we want to make sure this is satisfied for all the basis functions of  $S_n$ . So to compute  $v_n$ , suppose  $\{\phi_1, \dots, \phi_{n-1}\}$  is a basis for  $S_n$  ( $n-1$  basis functions because of the Dirichlet boundary conditions), then we need to find  $v_n \in S_n$  such that

$$a(v_n, \phi_i) = (f, \phi_i), \quad i = 1, 2, \dots, n-1$$

Now since  $v_n \in S_n$ , we write it as a linear combination of our basis functions:

$$v_n = \sum_{j=1}^{n-1} u_j \phi_j(x)$$

where now all we need to determine is  $u_1, \dots, u_{n-1}$ . We now have a linear system which we need to solve. To determine the matrix formulation, we see for  $i = 1, 2, \dots, n-1$ :

$$\begin{aligned} a(v_n, \phi_i) &= (f, \phi_i) \\ a\left(\sum_{j=1}^{n-1} u_j \phi_j(x), \phi_i\right) &= (f, \phi_i) \\ \sum_{j=1}^{n-1} a(\phi_j, \phi_i) u_j &= (f, \phi_i) \end{aligned}$$

Therefore, the coefficients  $u_1, \dots, u_{n-1}$  satisfy the linear system

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

where

$$K_{ij} = a(\phi_j, \phi_i) = \int_0^\ell \left[ \kappa(x) \frac{d\phi_j}{dx} \frac{d\phi_i}{dx} + p(x) \phi_j(x) \phi_i(x) \right] dx$$

and

$$\mathbf{f}_i = (f, \phi_i) = \int_0^\ell f(x) \phi_i(x) dx$$

(b) We can immediately see that  $K_{13} = 0$  as  $\phi_3 \phi_1 = 0$  and  $\frac{d\phi_3}{dx} \frac{d\phi_1}{dx} = 0$ , so we are happy. To calculate  $K_{12} = a(\phi_2, \phi_1) = \int_0^\ell \left[ \frac{d\phi_2}{dx} \frac{d\phi_1}{dx} + 3\phi_2(x) \phi_1(x) \right] dx$ , we first focus on  $\int_0^\ell \frac{d\phi_2}{dx} \frac{d\phi_1}{dx} dx$  and see

$$\int_0^\ell \frac{d\phi_2}{dx} \frac{d\phi_1}{dx} dx = \int_h^{2h} \frac{1}{h} \left( -\frac{1}{h} \right) dx = -\frac{1}{h} = -\frac{n}{\ell}$$

And we now look at  $\int_0^\ell 2\phi_2(x) \phi_1(x) dx$ . We see this is equal to

$$\begin{aligned} & 2 \int_h^{2h} \left( \frac{1}{h}(x-h) \right) \left( -\frac{1}{h}(x-2h) \right) dx \\ &= -\frac{2}{h^2} \int_h^{2h} (x-h)[(x-h)-h] dx \\ &= -\frac{2}{h^2} \int_h^{2h} [(x-h)^2 - h(x-h)] dx \\ &= -\frac{2}{h^2} \left[ \frac{1}{3}(x-h)^3 - h \frac{(x-h)^2}{2} \right]_h^{2h} \\ &= -\frac{2}{h^2} \left[ \frac{h^3}{3} - \frac{h^3}{2} \right] \\ &= \frac{h}{3} \\ &= \frac{\ell}{3n} \end{aligned}$$

and so we see  $K_{12} = -\frac{n}{\ell} + \frac{\ell}{3n}$ .

5. (20 points) *Fourier Series*. Given a thin bar of length  $\ell = \pi$  cm with  $\rho = 1$  g/cm<sup>3</sup>,  $c = 1$  J/gK, and  $\kappa = 1$  W/cmK, assume that the bar is initially at a uniform temperature of 0°C. Then, beginning at time zero, heat is added at a rate of  $f(x, t) = \sin(x)$  while the ends of the bar are held constant at 0°C.

- (a) Set up the corresponding initial-boundary-value problem.  
 (b) Solve this problem using Fourier series methods. Determine the steady-state solution by taking a limit. Draw a rough sketch of how the solution will evolve with time. (*Note*: you must show how to derive the solution from first principles and may not make use of a memorized formula.) (*Hint*: The solution has a relatively simple form.)

**Solution:**

(a) Using the information above with appropriate units (degrees Celsius, Watts per cubic centimeter, etc.), we obtain

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \sin(x), & 0 < x < \pi, t > 0 \\ u(x, 0) &= 0, & 0 < x < \pi \\ u(0, t) = u(\pi, t) &= 0, & t \geq 0. \end{aligned}$$

(b) Since the  $x$ -derivative part of this equation corresponds to the  $L_D$  operator, we expect the solution to have the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} a_n(t) \sin(nx).$$

Substituting this expression into the PDE above, we obtain

$$\sum_{n=1}^{\infty} \left[ \frac{da_n}{dt} + n^2 a_n(t) \right] \sin(nx) = \sin(x).$$

Since  $\sin(x)$  is already an eigenfunction, we do not need to expand the right hand side as a Fourier series. By orthogonality of the eigenfunctions, we have

$$\begin{aligned} \frac{da_1}{dt} + a_1(t) &= 1 \\ \frac{da_n}{dt} + n^2 a_n(t) &= 0, \quad \text{for } n = 2, 3, \dots \end{aligned}$$

Since the initial condition is equal to zero, we know that  $a_1(0) = 0$  and  $a_n(0) = 0$  for  $n = 2, 3, \dots$

We solve these ODEs using integrating factors. For  $n = 1$ , we obtain

$$\frac{d}{dt} [e^t a_1(t)] = e^t.$$

Integrating from 0 to  $t$  yields

$$e^t a_1(t) = e^t - 1 \quad \rightarrow \quad a_1(t) = 1 - e^{-t}.$$

Similarly, for  $n = 2, 3, \dots$ ,

$$\frac{d}{dt} [e^{n^2 t} a_n(t)] = 0,$$

which implies that  $a_n(t) = 0$  for  $n = 2, 3, \dots$ . Therefore, the solution is

$$u(x, t) = (1 - e^{-t}) \sin(x).$$

We obtain the steady-state solution  $u_s(x)$  by taking a limit as  $t \rightarrow \infty$ :

$$u_s(x) = \lim_{t \rightarrow \infty} u(x, t) = \lim_{t \rightarrow \infty} (1 - e^{-t}) \sin(x) = \sin(x).$$

Of course, this limit can be verified by means of the steady-state differential equation:

$$-\frac{d^2 u_s}{dx^2} = \sin(x), \quad u_s(0) = u_s(\pi) = 0.$$

Clearly,  $u_s$  solves this equation.

Therefore, as time increases from zero, the temperature distribution in the bar always maintains the shape of a sine function with amplitude given by  $1 - e^{-t}$ . Your sketch should show the temperature beginning at zero (a constant function equal to zero), which then grows as a sine curve with amplitude growing from zero to one as time goes to infinity.