CAAM 336

DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 2

Posted 28 April 2006.

For graduating students: due 12 noon on Thursday, 4 May 2006.
For non-graduating students: due 5pm on Wednesday, 10 May 2006.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are four questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   Show all your work to maximize partial credit.
5. Print your name on the line below:

   ____

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

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7. Staple this page to the front of your exam.

1. [25 points]
On the last two homework assignments we considered the diffusion and wave equations posed on a square domain in two dimensions. For this problem you will analyze the heat equation in two dimensions,
$$\frac{\partial u}{\partial t}(x,y,t) = \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t)$$
for $t \geq 0$, $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with boundary conditions $u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$ and initial condition $u(x,y,0) = \psi(x,y)$. This equation describes the flow of heat in a square plate.

For this problem you may use the fact that the symmetric operator L defined by
$$Lu = -\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big)$$
for $u(x,y)$ with homogeneous Dirichlet boundary conditions has the orthonormal eigenfunctions
$$\phi_{j,k} = 2\sin(j\pi x)\sin(k\pi y),$$
for positive integers j and k.

(a) What is the eigenvalue $\lambda_{j,k}$ of L associated with the eigenfunction $\phi_{j,k}$?

(b) We wish to write the solution to the heat equation in the form
$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\phi_{j,k}(x,y).$$
Derive an ordinary differential equation that describes $da_{j,k}/dt$. Be sure to specify how one obtains the initial condition, i.e., the value of $a_{j,k}(0)$.

(c) Write down the solution of the ordinary differential equation in (b) and use this to write a formula for $u(x,y,t)$.

(d) Describe the behavior of $u(x,y,t)$ as $t \to \infty$. (Be as precise as possible: don't simply say 'decays to zero' or 'blows up', but specify the dominant shape that $u(x,y,t)$ takes as $t \to \infty$.)

(e) Suppose you are given the function $v(x,y) = x$, which satisfies
$$v(x,0) = x, \quad v(x,1) = x, \quad v(0,y) = 0, \quad v(1,y) = 1.$$
Explain how you can use v to obtain a solution of the two-dimensional heat equation with inhomogeneous Dirichlet boundary conditions:
$$\frac{\partial u}{\partial t}(x,y,t) = \frac{\partial^2 u}{\partial x^2}(x,y,t) + \frac{\partial^2 u}{\partial y^2}(x,y,t)$$
for $t \geq 0$, $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with $u(x,y,0) = \psi(x,y)$ and
$$u(x,0,t) = x, \quad u(x,1,t) = x, \quad u(0,y,t) = 0, \quad u(1,y,t) = 1.$$

2. [25 points]
Consider the wave equation
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$
for $0 \leq x \leq 1$ and $t \geq 0$ with homogeneous Neumann boundary conditions,
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0$$
and initial data
$$u(x,0) = \psi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \gamma(x).$$

(a) Compute all the eigenvalues λ_k and eigenfunctions ϕ_k of the operator
$$Lu = -\frac{d^2 u}{dx^2}$$
for $0 \leq x \leq 1$ subject to the homogeneous Neumann boundary conditions
$$\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(1) = 0.$$

(b) Suppose we wish to write the solution to the wave equation in the form
$$u(x,t) = \sum_k a_k(t)\phi_k(x).$$
Write down an ordinary differential equation for the coefficients $a_k(t)$; be sure to specify the initial conditions for $a_k(0)$.

(c) Solve the ordinary differential equations from part (b). Please explain any special cases.

(d) Contrast the behavior of $u(x,t)$ as $t \to \infty$ with that of solutions to the one-dimensional heat equation with the same boundary conditions.

3. [20 points]
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} -500 & 500 \\ 500 & -500 \end{pmatrix}.$$

(a) Using the Fredholm Alternative as a guide, describe all vectors $\mathbf{b}$ for which the equation $\mathbf{Ax} = \mathbf{b}$ has a unique solution $\mathbf{x}$. When no unique solution $\mathbf{x}$ exists, are there (1) no solutions or (2) infinitely many solutions? If the latter, explain how to construct them.

(b) Compute the matrix exponential $e^{t\mathbf{A}}$ for this matrix and describe its behavior as $t \to \infty$.

(c) Write down the forward Euler and backward Euler methods for approximating the solution of the differential equation

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{A}\mathbf{y}(t).$$

(d) For the matrix $\mathbf{A}$ given above, what are the largest values of the time step Δt for which the approximate solutions produced by the forward and backward Euler methods will mimic the behavior of the true solution as $t \to \infty$?

4. [30 points]
Let ε denote some constant, and consider the differential equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) - \varepsilon u(x,t) = f(x,t).$$

for $0 \le x \le 1$, $t \ge 0$, with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = \psi(x).$$

Describe how to solve this differential equation using the finite element method. Provide as much detail as you can about the following points.

(a) What is the weak form of this differential equation?

(b) State the Galerkin problem based on the approximating subspace $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$ for linearly independent basis functions $\phi_1, \ldots, \phi_N$.

(c) Show how this problem leads to a differential equation of the form

$$\mathbf{A}\frac{d}{dt}\mathbf{a}(t) + \mathbf{B}\mathbf{a}(t) = \mathbf{f},$$

where you should specify the entries of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{f}$.

(d) Describe how to solve this equation using the backward Euler method.

(e) Does the solution you obtain depend on the choice of the basis functions $\phi_1, \ldots, \phi_N$?

(f) Describe how your method would change if the equation had inhomogeneous Dirichlet boundary conditions,

$$u(0,t) = a, \qquad u(1,t) = b.$$