There was a bit of confusion with problem #2 on the exam. This was the problem that went

Find a Fourier series solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $0 \leq x \leq 2$ and $t \geq 0$ with inhomogeneous Dirichlet boundary conditions,

$$u(0,t) = 0, \quad u(2,t) = 3$$

and initial data

$$u(x,0) = 0$$
$$\frac{\partial u}{\partial t}(x,0) = 0.$$

(A note: The initial and boundary conditions are not compatible here- just ignore this and proceed as if they are compatible.)

The confusion stemmed from the fact that we ask you to solve a problem where $u(2,t) = 3$ but $u(x,0) = 0$. The big question is, what the heck happens at $(0,0)$?? Is it 0 or 3?

I will go through the solution of the problem, highlighting the common mistakes, and illustrating the solution and how these weird conditions fit in to it.

First things first, there are a whole lot of zeros here. The only thing that's nonzero is the boundary condition at $x = 2$. So we have inhomogeneous boundary conditions. The thing to do here is shift the problem so that we have homogeneous boundary conditions. To do this, introduce a function that matches the boundary condition. The easiest one I can think of is

$$p(x) = \frac{3}{2}x.$$

We then shift everything, introducing a new function v:

$$v(x,t) = u(x,t) - p(x).$$

When you do this, and this is the most common mistake made on this problem, you must filter this shift through EVERYTHING in the problem, including the initial conditions. Let's do this now.

$$\begin{aligned}
\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} &= \frac{\partial^2 (u-p)}{\partial t^2} - \frac{\partial^2 (u-p)}{\partial x^2} \\
&= \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} \\
&= 0.
\end{aligned}$$

The difference of the derivatives of u are zero because it solves the original differential equation. Two derivatives of a linear function are zero no matter what, so the p derivatives here are zero. Moving on to the boundary conditions, we have

$$\begin{aligned} v(0,t) &= u(0,t) - p(0) = 0 - 0 = 0, \\ v(2,t) &= u(2,t) - p(2) = 3 - 3 = 0. \end{aligned}$$

So everything is cool and we have homogeneous boundary conditions. Now for the last part, the part that many people forgot. Shifting the initial conditions.

$$\begin{aligned} v(x,0) &= u(x,0) - p(x) = 0 - \frac{3}{2}x = -\frac{3}{2}x \\ \frac{\partial v}{\partial t}(x,0) &= \frac{\partial u}{\partial t}(x,0) - \frac{\partial p}{\partial t}(x,0) = 0 - 0 = 0. \end{aligned}$$

In the initial condition on the derivative, you must remember that p is a function only of x, so its t derivative is automatically zero.

Okay, so now we need to find v. We have Dirichlet boundary conditions going from 0 to 2, so the eigenfunctions are

$$\phi_n(x) = \sin\left(\frac{n\pi}{2}x\right),$$

with corresponding eigenvalues

$$\lambda_n = \frac{n^2\pi^2}{4}.$$

We expect the solution to take on the form

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\left(\frac{n\pi}{2}x\right).$$

Plug this in to the differential equation to get

$$\sum_{n=1}^{\infty} a_n''(t)\sin\left(\frac{n\pi}{2}x\right) + \sum_{n=1}^{\infty} a_n(t)\frac{n^2\pi^2}{4}\sin\left(\frac{n\pi}{2}x\right) = 0.$$

Take the inner produce with one of the eigenfunctions and simplifying gives the ODE

$$a_n''(t) + \frac{n^2\pi^2}{4}a_n(t) = 0.$$

We'd like to solve this, but what about those pesky initial conditions? Well, we know that

$$v(x,0) = -\frac{3}{2}x = \sum_{n=1}^{\infty} a_n(0)\sin\left(\frac{n\pi}{2}x\right),$$

and

$$\frac{\partial v}{\partial t}(x,0) = 0 = \sum_{n=1}^{\infty} a_n'(0)\sin\left(\frac{n\pi}{2}x\right).$$

The second equation is simple. It tells us that $a_n'(0) = 0$. As for the first one, we have to find the Fourier coefficients. Take the inner product, obtaining

$$\begin{aligned}
a_n(0) &= \frac{2}{2}\int_0^2 -\frac{3}{2}x \sin\left(\frac{n\pi}{2}x\right)\,dx \\
&= -\frac{3}{2}\int_0^2 x\sin\left(\frac{n\pi}{2}x\right)\,dx \\
&= -\frac{3}{2}\left[-\frac{2}{n\pi}x\cos\left(\frac{n\pi}{2}x\right) - \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}x\right)\right]_{x=0}^2 \\
&= -\frac{3}{2}\left[-\frac{4}{n\pi}\cos(n\pi)\right] \\
&= \frac{6(-1)^n}{n\pi}.
\end{aligned}$$

So now we have the initial equations for the ODE. We can solve it, obtaining

$$a_n(t) = \frac{6(-1)^n}{n\pi}\cos\left(\frac{n\pi}{2}t\right).$$

With a_n found, we then have v:

$$v(x,t) = \sum_{n=1}^{\infty} \frac{6(-1)^n}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\sin\left(\frac{n\pi}{2}x\right).$$

Shifting the problem back, we obtain the solution to the original differential equation:

$$u(x,t) = \frac{3}{2}x + \sum_{n=1}^{\infty} \frac{6(-1)^n}{n\pi}\cos\left(\frac{n\pi}{2}t\right)\sin\left(\frac{n\pi}{2}x\right).$$

But what does this mean for those funky initial and boundary conditions? Well, the best thing to do is to look at a graph of the solution and see what we have. I fire up Maple, toss in the preliminary

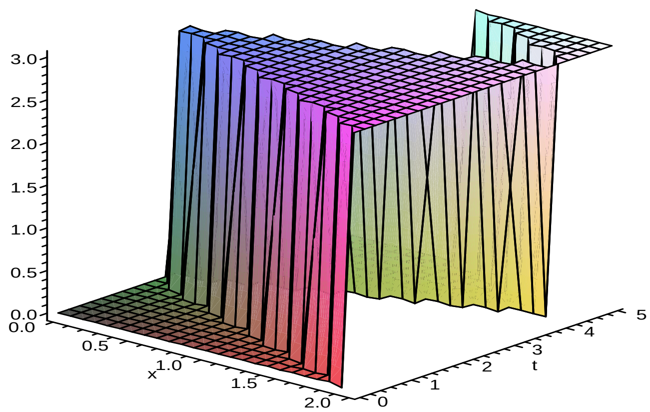
```
> with(plots):
> assume(n,integer):
```

Now I define the coefficients.

```
> a:=n->6*(-1)^n/(n*Pi);
```

I want to make a 3d plot to get an idea of what it looks like from the different angles all at once.

```
> plot3d(3*x/2+sum(a(n)*cos(n*Pi*t/2)*sin(n*Pi*x/2),n=1..500),x=0..2,t=0..5);
```

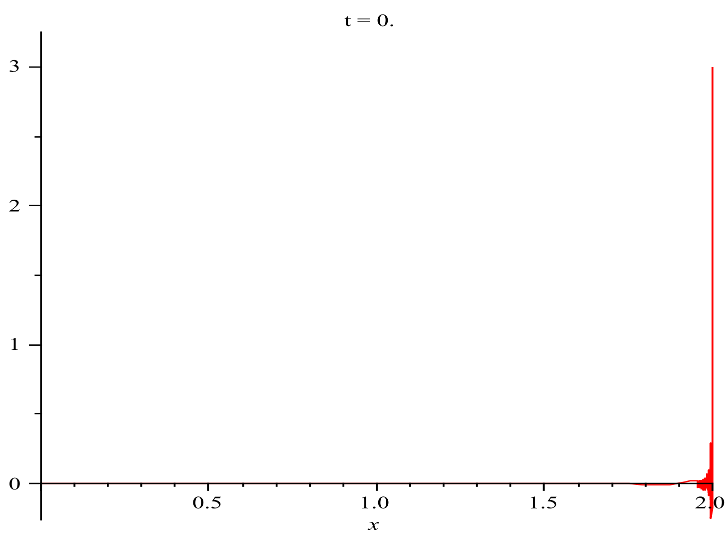


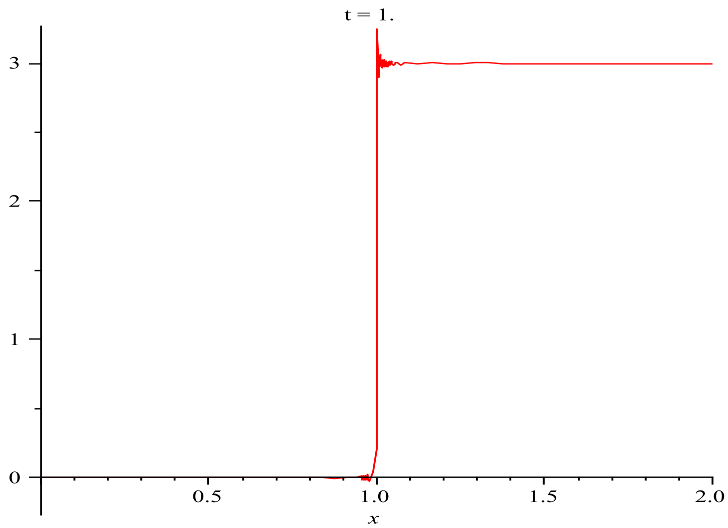
Notice the sharp cliff. At time $t = 0$, it does look like everything is zero, then over at the corner $(2, 0)$, things jump to 3. It then looks like the cliff moves back and forth as time goes on.

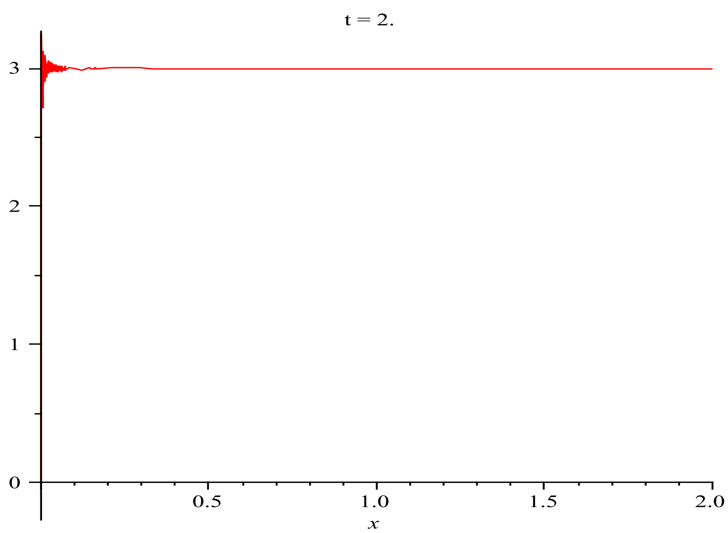
This drawing does show how it looks at the weird inconsistency, but this is the wave equation, so it moves. So why not animate it?

```
>   animate(plot,[3*x/2+sum(a(n)*cos(n*Pi*t/2)*sin(n*Pi*x/2),n=1..1000),x=0..2],t=0..4);
```

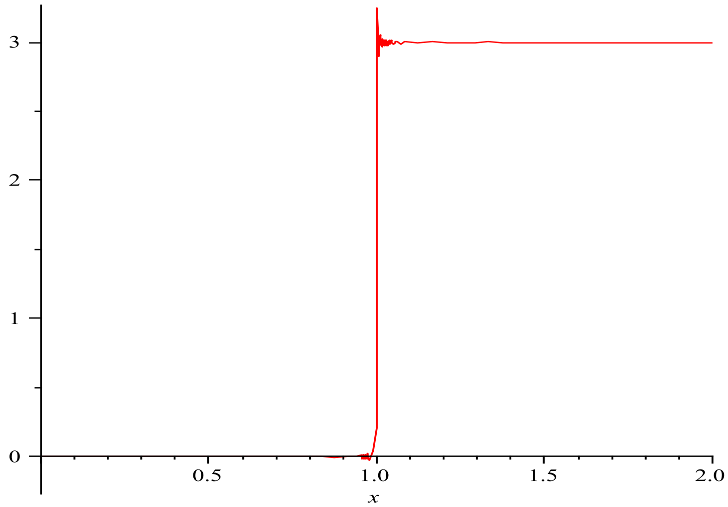
When you run this, you see what you expect, everything is zero at time zero. However, there is a bit of strangeness off at the end. That's from the Gibbs phenomenon where the Fourier series is trying to be two places at once (more or less). Let it run, and you see the "cliff" of height 3 move off to the left, hit $x = 0$, then go back. This is how the Fourier series deals with the incompatibility.




t = 1.
3
2
1
0
0.5
1.0
1.5
2.0
x


t = 2.
3
2
1
0
0.5
1.0
1.5
2.0
x

t = 3.



t = 4.