# CAAM 336

# DIFFERENTIAL EQUATIONS IN SCIENCE AND ENGINEERING

Examination 1

Posted 19 October 2006, 10:00 AM.

Due 12:00 noon on Thursday, 26 October 2006.

Instructions:

1. Time limit: **4 uninterrupted hours**.
2. There are five questions worth a total of 100 points.
   Please do not look at the questions until you begin the exam.
3. You *may not* use any outside resources, such as books, notes, problem sets, friends, calculators, or MATLAB.
4. Please answer the questions thoroughly and justify all your answers.
   If in doubt, provide more detail rather than less.
   *Show all your work to maximize partial credit.*
5. Print your name on the line below:

   ________________________________________

6. Indicate that this is your own individual effort in compliance with the instructions above and the honor system by writing out in full and signing the traditional pledge on the lines below.

   ________________________________________

   ________________________________________

   ________________________________________

7. Staple this page to the front of your exam.

1. [20 points]

   (a) Determine if the following sets are vector spaces or not- justify your answer. (Define addition and scalar multiplication in the obvious way.)

      i. $\{f \in C[0,1] : \int_0^1 f(x)dx = 0\}$

      ii. The set of all vectors in $\mathbf{R}^3$, $(x_1, x_2, x_3)$, such that $x_1 + 2x_2 + 3x_3 = 4$.

      iii. The set of all vectors in $\mathbf{R}^2$, $(x_1, x_2)$, such that $x_1 \geq 0$.

      iv. $\{f \in C^2[0,1] : -\frac{d^2 f}{dx^2} = 0\}$

   (b) Determine if the following operators are linear or not. Justify your answer.

      i. $L : C^2[0,1] \to C[0,1]$ defined by

$$Lu := -\frac{d^2u}{dx^2} + (\sin x)u$$

      ii. $L : C^2[0,1] \to C[0,1]$ defined by

$$Lu := -\frac{d^2u}{dx^2} + \frac{du}{dx}u$$

   (c) Recall that we defined $C_D^2[0,1] = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$. Define the operator $L : C_D^2[0,1] \to C[0,1]$ by

$$Lu := -\frac{d^2u}{dx^2} - \pi^2 u.$$

   Find the null space of $L$, $N(L)$.

2. [20 points]
Consider the matrix and vector

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}.$$

(a) Compute the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Solve $\mathbf{Ax} = \mathbf{b}$ by using the spectral method.

3. [20 points]

(a) Find the nearest point to $\mathbf{v} = (1, 2, 3) \in \mathbf{R}^3$ from the subspace with basis $\{(1, 0, 1), (1, -1, 1)\}$.

(b) The vector space $\mathcal{P}_1$, the set of all polynomials of degree $\leq 1$, is a finite dimensional subspace of $\mathcal{V} = C[0, 1]$. With the inner product $(u, v) = \int_0^1 u(x)v(x)\,dx$, find the best approximation to $f(x) = x^2$ from $\mathcal{P}_1$.

(c) Using the same inner product as above, show that $\{1, x - 1/2\}$ is an orthogonal basis for $\mathcal{P}_1$.

(d) Using the orthogonal basis above, again find the best approximation to $f(x) = x^2$. Check that you get the same answer as in part (b).

4. [20 points]
Consider the problem

$$-\frac{d^2u}{dx^2} + u = f(x)$$

$$u(0) = 0 \qquad \frac{du}{dx}(1) = 0.$$

(a) Set the problem up as a linear operator equation. Be careful to define the domain space correctly.

(b) Find all of the eigenvalues and eigenfunctions of the operator.

(c) Express the solution for general $f$ as an infinite series. Be as explicit as possible without knowing $f$.

(d) Find the solution when $f(x) = 7\sin(\pi x/2) - 3\sin(9\pi x/2)$.

5. [20 points]
   Consider the equation,

$$-\frac{d}{dx}\Big(\kappa(x)\frac{du}{dx}(x)\Big) + p(x)u(x) = f(x), \qquad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0,$$

Here $\kappa(x)$ and $p(x)$ are positive-valued functions.

(a) State what it means for a linear operator to be *symmetric*, and show that the operator $L : C_D^2[0,1] \to C[0,1]$, defined by

$$Lu = -\frac{d}{dx}\Big(\kappa(x)\frac{du}{dx}(x)\Big) + p(x)u, \qquad 0 < x < 1,$$

is a symmetric operator acting on the space

$$C_D^2[0,1] = \Big\{u \in C^2[0,1] : u(0) = u(1) = 0\Big\}.$$

(b) Derive the weak form of this equation with the above boundary conditions, i.e., derive the weak problem

$$a(u,v) = (f,v), \quad \text{for all } v \in C_D^2[0,1].$$

Specify the bilinear form $a(u,v)$, and show that it is an inner product on $C_D^2[0,1]$.

(c) Suppose that $V_N = \text{span}\{\phi_1, \ldots, \phi_N\}$ is an $N$-dimensional subspace of $C_D^2[0,1]$. (Do not assume a particular form for the functions $\phi_1, \ldots, \phi_N$ at this point.) Show how the Galerkin problem

$$a(u_N, v) = (f, v), \quad \text{for all } v \in V_N$$

leads to the linear system $\mathbf{Ku} = \mathbf{f}$. Be sure to specify the entries of $\mathbf{K}$, $\mathbf{u}$, and $\mathbf{f}$.

(d) Now suppose we take for $\phi_1, \ldots, \phi_N$ the standard piecewise linear 'hat' functions on the uniform mesh $h = 1/N + 1$, $x_k = kh$,

$$\phi_k(x) = \begin{cases} (x - x_{k-1})/h, & x \in [x_{k-1}, x_k); \\ (x_{k+1} - x)/h, & x \in [x_k, x_{k+1}); \\ 0, & \text{otherwise.} \end{cases}$$

Let $\kappa(x) = 1$ and $p(x) = 2$. Compute $\mathbf{K}_{ii}$. What can you say about the structure of $\mathbf{K}$ and why?