CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 12

Posted Wednesday 29 November 2006. Due Friday 8 December 2006, IN CLASS.

1. [50 points]
   In this problem we wish to study the diffusion equation in two dimensions. In place of the one dimensional equation, $-d^2u/dx^2 = f$, we now have

$$-\Big(\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)\Big) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1,$$

   with homogeneous Dirichlet boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$ for all $0 \le x \le 1$ and $0 \le y \le 1$. The associated operator L, defined as

$$Lu = -\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big),$$

   acting on the space $C_D^2[0,1]^2$ consisting of twice continuously differentiable functions on $[0,1] \times [0,1]$ with homogeneous boundary conditions, is symmetric and positive definite. We can solve the differential equation $Lu = f$ using the spectral method just as we have seen in class before. This problem will walk you through the process; you may consult Section 8.2 of the text for hints.

   (a) Verify that the functions

$$\phi_{j,k}(x,y) = \sin(j\pi x)\sin(k\pi y)$$

   are eigenfunctions of L for $j,k = 1,2,\ldots$.
   (To do this, you simply need to show that $L\phi_{j,k} = \lambda_{j,k}\phi_{j,k}$ for some constant $\lambda_{j,k}$.)

   (b) What is the eigenvalue $\lambda_{j,k}$ associated with $\phi_{j,k}$?

   (c) Compute the inner product $(\phi_{j,k}, \phi_{j,k})$, where

$$(v,w) = \int_0^1 \int_0^1 v(x,y)w(x,y)\,dx\,dy.$$

   (d) Let $f(x,y) = x(1-y)$. Compute the inner product $(f, \phi_{j,k})$.

   (e) The solution to the diffusion equation is given by the spectral method,

$$u(x,y) = \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{(f,\phi_{j,k})}{\lambda_{j,k}(\phi_{j,k},\phi_{j,k})}\phi_{j,k}(x,y).$$

   In MATLAB plot the partial sum

$$u_{10}(x,y) = \sum_{j=1}^{10}\sum_{k=1}^{10} \frac{(f,\phi_{j,k})}{\lambda_{j,k}(\phi_{j,k},\phi_{j,k})}\phi_{j,k}(x,y).$$

   Hint: If you wanted to plot $\phi_{1,1}(x,y) = \sin(\pi x)\sin(\pi x)$, you could use

```
x = linspace(0,1,20); y = linspace(0,1,20);
[X,Y] = meshgrid(x,y);
Phi11 = sin(pi*X).*sin(pi*Y);
surf(X,Y,Phi11)
```

please turn over...

2. [50 points]
In the first exercise, you solved the homogeneous diffusion equation on a square domain in two dimensions. In the present exercise you will solve the wave equation on the same domain, a model of a vibrating membrane stretched over a square frame—that is, a square drum:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

with $0 \le x \le 1$, and $0 \le y \le 1$, and $t \ge 0$. Take homogeneous Dirichlet boundary conditions

$$u(x,0,t) = u(x,1,t) = u(0,y,t) = u(1,y,t) = 0$$

for all x and y such that $0 \le x \le 1$ and $0 \le y \le 1$ and all $t \ge 0$, and consider the initial conditions

$$u(x,y,0) = \psi(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} b_{j,k}\phi_{j,k}(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0) = \gamma(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} d_{j,k}\phi_{j,k}(x,y).$$

Here the $\phi_{j,k}(x,y)$, $j,k \ge 1$ denote the eigenfunctions of the operator

$$Lu = -\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big),$$

with homogeneous Dirichlet boundary conditions given in Problem Set 11. You may use without proof that these eigenfunctions are orthogonal, and use the eigenvalues $\lambda_{j,k}$ computed in problem 1.

(a) We wish to write the solution to the wave equation in the form

$$u(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}(t)\phi_{j,k}(x,y).$$

Show that the coefficients $a_{j,k}(t)$ obey the ordinary differential equation

$$\frac{d^2 a_{j,k}}{dt^2}(t) = -\lambda_{j,k} a_{j,k}(t)$$

with initial conditions

$$a_{j,k}(0) = b_{j,k}, \qquad \frac{da_{j,k}}{dt}(0) = d_{j,k}.$$

(b) Write down the solution to the differential equation in part (a).

(c) Use your solution to part (b) to write out a formula for the solution $u(x,y,t)$.

(d) Suppose that the initial conditions are

$$\psi(x,y) = 300xy(1-x)(1-y)^2, \qquad \gamma(x,y) = 0.$$

Modify the `wave2d.m` program from the class web site to plot the solution as a function of time. Submit plots of your solution at times $t = 0$, $\sqrt{2}/2$, $\sqrt{2}$, $3\sqrt{2}/2$, $2\sqrt{2}$.
(Note that we have not normalized these eigenfunctions; as seen in problem 1, $(\phi_{j,k}, \phi_{j,k}) = 1/4$. You will need to take this into account in your formula for $b_{j,k}$.)