CAAM 336 · DIFFERENTIAL EQUATIONS

Problem Set 9

Posted Wednesday 21 March 2006. Due Wednesday 28 March 2006, in class.

1. [36 points]
For each of the following examples, (1) compute the matrix exponential $e^{t\mathbf{A}}$ by hand, and (2) explain the behavior of $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$ as $t \to \infty$ given that $\mathbf{x}(0) = (2,\ 0)^T$.

$$(a)\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (b)\quad \mathbf{A} = \begin{pmatrix} -50 & 49 \\ 49 & -50 \end{pmatrix} \qquad (c)\quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Hint for part (c): This matrix is not symmetric, but it does have a decomposition of of the form $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^*$, where $\mathbf{U}$ is a matrix of (possibly complex) eigenvectors, $\mathbf{D}$ is a diagonal matrix of eigenvalues, and $\mathbf{U}^*$ is the transpose of the complex conjugate of $\mathbf{U}$.

2. [30 points]
For the matrix $\mathbf{A}$ from problem 1(b), describe how large one can choose the time step Δt so that the forward Euler method applied to $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$,

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{A}\mathbf{x}_k,$$

will produce a solution that qualitatively matches the behavior of the true solution (i.e., the approximations $\mathbf{x}_k$ should grow, decay, or remain of the same size as the true solution does).

Answer the same question for the backward Euler method

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{A}\mathbf{x}_{k+1}.$$

3. [30 points: 6 points for (a), 12 pts each (b) and (c)]
There exist a host of alternatives to the forward and backward Euler methods for approximating the solution of the differential equation $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$. For example, the *trapezoid rule* method has the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \tfrac{1}{2}\Delta t \mathbf{A}(\mathbf{x}_k + \mathbf{x}_{k+1}),$$

where $\Delta t > 0$ is the time-step.

   (a) Using our discussion of the backward Euler method as a departure point, describe how to implement the trapezoid rule method to find $\mathbf{x}_{k+1}$ given $\mathbf{x}_k$. In particular, what linear system of algebraic equations needs to be solved at each step?

   (b) Consider the matrix and initial condition

$$\mathbf{A} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   Approximate the solution to $d\mathbf{x}/dt = \mathbf{A}\mathbf{x}$ on the interval $t \in [0, 1]$ for time step $\Delta t = .05$. Produce a `semilogy` plot showing $t = k\Delta t$ versus $\|\mathbf{x}_k\|$ over $t \in [0, 1]$. (Use the `norm` command in MATLAB.)

   (c) Produce a `loglog` plot showing Δt versus the error in the trapezoid rule and backward Euler approximations for the matrix and initial condition in part (b) at time $t = 1$. To compute the error, first find the exact solution $\mathbf{x}(1) = \mathrm{e}^{\mathbf{A}}\mathbf{x}(0)$ using the `expm` command, then compute the norms $\|\widehat{\mathbf{x}} - \mathbf{x}(1)\|$, where $\widehat{\mathbf{x}}$ denotes your approximation to $\mathbf{x}(1)$ from the trapezoid or implicit Euler methods. Start your plot with $\Delta t = 1/2$ and use sufficiently many smaller values of Δt to make the trend in your plot clear. Which method is better?