Math 126A Spring, 2001

Name: ____________________________

Kirk Kris

Section 1:30 2:30 1:30 2:30
(circle one) AA AC AB AD

Quiz Two

Show all of your work, and justify your answers. Answers without work or justification will not receive full credit. Leave your answers in exact form; do not try to convert expressions such as $\pi$ or $\sqrt{2}$ into decimal approximations. You may not use notes or calculators.

This quiz has four (4) problems. Be sure that you answer all of them.

1) (5 points) Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$$

is convergent or divergent (justify your answer).

2) (5 points) We are told that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{n^2 + 1}$$

is convergent, and that the sum of the first six terms is

$$\sum_{n=1}^{6} (-1)^{n-1} \frac{2}{n^2 + 1} = \frac{2}{1^2 + 1} - \frac{2}{2^2 + 1} + \frac{2}{3^2 + 1} - \frac{2}{4^2 + 1} + \frac{2}{5^2 + 1} - \frac{2}{6^2 + 1} \approx 0.70522.$$

Give an estimate for the error of this partial sum from the sum of the entire series.
3 (5 points) Determine whether the series

$$\sum_{n=1}^{\infty} \left( \frac{5n - 3}{4n + 1} \right)^n$$

is convergent or divergent (justify your answer).

4 (5 points) Determine for what values of $x$ the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n x^n}{2^n}$$

is convergent.
The series \( \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} \) is convergent. We use the alternating series test. Since the series is alternating, we must only show that \( \lim_{n \to \infty} \frac{\ln n}{n} = 0 \) and that the terms \( \frac{\ln n}{n} \) are decreasing. The first is an application of L'Hôpital's rule:

\[
\lim_{n \to \infty} \frac{\ln n}{n} = \lim_{n \to \infty} \frac{1/n}{1} = 0.
\]

That the terms are decreasing follows from looking at the derivative of the function \( f(x) = \frac{\ln x}{x} \).

The error (or remainder) of a partial sum in an alternating series is always bounded by the absolute value of the next term. Thus \( R_6 = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2 + 1} - \sum_{n=1}^{6} (-1)^{n-1} \frac{2}{n^2 + 1} = \sum_{n=7}^{\infty} (-1)^{n-1} \frac{2}{n^2 + 1} \) is bounded by the seventh term:

\[
|R_6| \leq \left| (-1)^{7-1} \frac{2}{7^2 + 1} \right| = \frac{1}{25} = 0.04.
\]

The series \( \sum_{n=1}^{\infty} \left( \frac{5n-3}{4n+1} \right)^n \) is divergent. Using the Root Test, we see that

\[
\sqrt[n]{a_n} = \left| \frac{5n-3}{4n+1} \right| \to \frac{5}{4}
\]

as \( n \to \infty \). Since \( 5/4 > 1 \), the Root Test implies that this series is divergent.

We could have used the Test for Divergence, and shown that \( |a_n| \to \infty \). This also shows that the series diverges.

The series \( \sum_{n=1}^{\infty} (-1)^n \frac{n2^n}{2^n} \) is convergent for \( |x| < 2 \), or for the interval \( (-2, 2) \). We see this by using the Ratio Test:

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1}(n+1)x^{n+1}/2^{n+1}}{(-1)^n n2^n/2^n} \right| = |x| \frac{n+1}{2n} \to |x|/2.
\]

Thus this series converges for \( |x|/2 < 1 \) and diverges for \( |x|/2 > 1 \). At \( x = 2 \), this series is \( \sum (-1)^n n \), which is divergent, and at \( x = -2 \), the series is \( \sum n \), which is also divergent. Thus the series converges for \( |x| < 2 \) and diverges for \( |x| \geq 2 \).