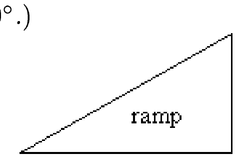
1. (10 points) Decide whether the following statements are true or false. (In **this** problem no explanation is needed. In **all** other problems you have to explain your answers!) Each answer is worth 2 points.

   1) The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is always a nonzero scalar. **F**
   The volume is zero if the vectors are co-planar.
   2) In three dimensions, $< 0, 0, 0 >$ is the *only* vector with magnitude 0. **T**
   This was mentioned in lecture and is on page 790.
   3) If **a** and **b** are coplanar, then $\mathbf{a} \cdot \mathbf{b} = 0$. **F**
   $\mathbf{a} \cdot \mathbf{b} = 0$ *if and only if* **a** *and* **b** *are orthogonal. Vectors that are coplanar are not necessarily orthogonal.*
   4) Two planes are orthogonal if their normal vectors are parallel. **F**
   Two vectors are orthogonal if their normal vectors are orthogonal.
   5) If **a** is parallel to **c**, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = 0$, where **a**, **b** and **c** are nonzero vectors. **F**
   Note, $\mathbf{b} \times \mathbf{c}$ *is orthogonal to* **a**. *The cross product is zero for parallel vectors.*

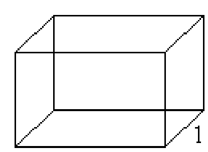
2. (10 points) Peter from One Guy and a Pick-up Truck Moving Company pushes a couch up a ramp that is at an angle α above the horizontal. The ramp is also 1 meter high and 1.5 meters long at the base as seen in the picture below. If the force that Peter exerts on the couch has a magnitude of 24 N and is at an angle of 30° above the horizontal, what is the work done? (*Note,* α *does not necessarily equal* 30°.)



Solution

$\mathbf{F} = \langle 24\cos 30°, 24 \sin 30° \rangle = \langle (24)\left(\frac{\sqrt{3}}{2}\right), (24)\left(\frac{1}{2}\right) \rangle = \langle 12\sqrt{3}, 12 \rangle$. $\mathbf{D} = \langle 1.5, 1 \rangle$. Thus $W = \mathbf{F} \cdot \mathbf{D} = 12\sqrt{3}(1.5) + 12(1) = 18\sqrt{3} + 12$. [Alternatively, the ramp has angle of $\alpha = \tan^{-1}(\frac{1}{1.5}) = \tan^{-1}(\frac{2}{3})$ above the horizontal. Since $\tan 30° = \frac{1}{\sqrt{3}} \approx \frac{1}{1.7} < \frac{1}{1.5} = \tan\alpha$, then $30° < \alpha$. Thus, angle between **F** and **D** is $\theta = \alpha - 30$, implying $W = |\mathbf{F}||\mathbf{D}|\cos(\alpha - 30) = 24\sqrt{3.25}\cos(\alpha - 30)$.]

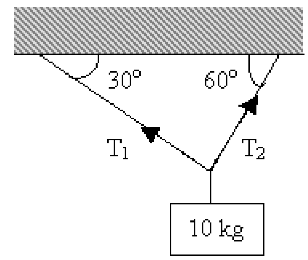
3. (10 points) A brick has dimensions $2 \times 1 \times 1$ as shown in the picture below. Find the angle between a diagonal of the brick with its longest edge. (*Finding* $\arccos\theta$ *or* $\arcsin\theta$ *will suffice as the answer.*)



Solution

(*This is similar to 12.3 #55.*) For convenience, consider consider a brick with the given dimensions positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. Further, assume its longest edge lies along the x-axis. The diagonal of the rectangle that begins at the origin and ends at (2,1,1) has vector representation $\mathbf{a} =< 2, 1, 1 >$. The angle between this vector and the vector of the longest edge which also begins at the origin and runs along the x-axis [this is, $\mathbf{b} =< 2, 0, 0 >$] is given by $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{4}{\sqrt{24}} = \frac{2}{\sqrt{6}}$. Hence, $\theta = cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$.

4. (20 points) A 10-kg weight hangs from two wires as shown below. Find the tensions (forces) $\mathbf{T_1}$ and $\mathbf{T_2}$ in both wires and their magnitudes.



Solution

Note, $\mathbf{T_1} =< -|\mathbf{T_1}|\cos 30°, |\mathbf{T_1}|\sin 30° >=< -\frac{\sqrt{3}}{2}|\mathbf{T_1}|, \frac{1}{2}|\mathbf{T_1}| >$ and $\mathbf{T_2} =< |\mathbf{T_2}|\cos 60°, |\mathbf{T_2}|\sin 60° >=< \frac{1}{2}|\mathbf{T_2}|, \frac{\sqrt{3}}{2}|\mathbf{T_2}| >$ We can consider the weight of the chain to be concentrated at its midpoint. This vector is given by $\mathbf{w} =< 0, -10g >=< 0, (-10)(9.8) >=< 0, -98 >$.

Since the system is in equilibrium, we must have $\mathbf{T_1} + \mathbf{T_2} = -\mathbf{w}$. Therefore, $-\frac{\sqrt{3}}{2}|\mathbf{T_1}| + \frac{1}{2}|\mathbf{T_2}| = 0$. So, $\sqrt{3}|\mathbf{T_1}| = |\mathbf{T_2}|$.

We also have $\frac{1}{2}|\mathbf{T_1}| + \frac{\sqrt{3}}{2}|\mathbf{T_2}| = 98$. Substituting our previous result yields that $|\mathbf{T_1}| + 3|\mathbf{T_1}| = 196$.

We have our result: $|\mathbf{T_1}| = 49, |\mathbf{T_2}| = 49\sqrt{3}, \mathbf{T_1} =< -\frac{\sqrt{3}}{2}49, \frac{49}{2} >, \mathbf{T_2} =< \frac{\sqrt{3}}{2}49, \frac{147}{2} >$.

5. (30 points). A roller coaster is designed to follow a path along a curve C given by the parametric equations:

$$x = \cos^2 t, \quad y = \sin t, \quad 0 \le t \le \pi.$$

1) (10 points) Find dy/dx.
2) (5 points) Sketch C.
3) (10 points) Find the area bounded by C and the lines $x = 1$ and $y = 1$.
4) (5 pts) Write down, but do not evaluate, an integral that represents the length of C.

Solution

1) First, $dx/dt = -2\cos t \sin t, dy/dt = \cos t$. Therefore, $dy/dx = (dy/dt)/(dx/dt) = -1/(2\sin t)$ provided $dx/dt = -2\cos t \sin t \ne 0$. That is, $t \ne 0, \pm\frac{\pi}{2}, \pm\pi, \ldots$.
2) Sketch is $x = 1 - y^2$ in the first quadrant.
3) Area of region = (area of 1 by 1 square) − (area under C and above x-axis). The area under C and above x-axis $= \int_0^1 y dx = \int_{\pi/2}^0 \sin t(-2\cos t \sin t)\ dt = -2\int_{\pi/2}^0 \sin^2 t \cos t\ dt =_{u=\sin t} -2\int_1^0 u^2\ du = -2\frac{u^3}{3}|_1^0 = -\frac{2}{3}(0^3 - 1^3) = \frac{2}{3}$. Thus, area of region $= 1 - \frac{2}{3} = \frac{1}{3}$. [Alternatively, area $= \int_0^1 (1-y)dx = \int_{\pi/2}^0 (1-\sin t)(-2\cos t \sin t)\ dt = \frac{1}{3}$.]
4) C is traversed once for $0 \le t \le \pi/2$, thus its arclength is $\int_0^{\pi/2} \sqrt{(-2\cos t \sin t)^2 + (\cos t)^2}\quad dt = \int_0^{\pi/2} \sqrt{(4\sin^2 t\ + 1)\cos^2 t}\ \ dt = \int_0^{\pi/2} \cos t \sqrt{4\sin^2 t\ + 1}\ \ dt$.

6. (20 points) Consider the line $L: -x = \frac{y}{3} = \frac{z}{4}$ and the two points $A(1,0,0)$, $B(0,1,0)$.

1) (5 points) Find parametric equations for the line through A and parallel to L.
2) (10 points) Find an equation for the plane that contains B and the line found in part (1).
3) (5 points) Notice that L is parallel to the plane found in part (2). Find the distance from L to this plane.

Solution

1) L has direction $\mathbf{v} = \langle -1, 3, 4\rangle$, so the line also has $\mathbf{v}$ as its direction. Then the line has parametric equations: $x = 1 - t, y = 3t, z = 4t$.

2) The plane contains A and B. A third point in the plane can be found by taking any other point on the line found in part (1). In particular, taking $t = 1$ gives the point $C(0, 3, 4)$. Then, a normal vector for the plane is $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 3 & 4 \end{vmatrix} = 4\mathbf{i} - (-4)\mathbf{j} + (-2)\mathbf{k} = \langle 4, 4, -2\rangle$. Since A is in the plane, an equation for the plane is $\langle 4, 4, -2\rangle \cdot \langle x - 1, y, z\rangle = 0$ or, equivalently, $4x + 4y - 2z - 4 = 0$.

3) The distance from L to this plane equals the distance from any point in L to the plane. One point in L is $O(0, 0, 0)$. Its distance to the plane is given by $|\text{comp}_{\mathbf{n}}\overrightarrow{AO}| = \frac{|\mathbf{n}\cdot\overrightarrow{AO}|}{|\mathbf{n}|} = \frac{|\langle 4,4,-2\rangle\cdot\langle -1,0,0\rangle|}{\sqrt{4^2+4^2+(-2)^2}} = \frac{4}{\sqrt{36}} = \frac{2}{3}$.