

1 (10 points) For the following sequences and series, state whether they converge or diverge. No work is necessary, and no partial credit will be given.

(a) (2 points) $\{\sin 2n\}_{n=1}^{\infty}$. Diverges. Does not settle down to anything as $n \rightarrow \infty$.

(b) (2 points) $\left\{\frac{n^2 + 5n}{2n^2 + 7}\right\}_{n=1}^{\infty}$. Converges. Just take the limit, the limit is $\frac{1}{2}$.

(c) (2 points) $\sum_{n=0}^{\infty} e^{-n}$. Converges. This is a geometric series with $r = \frac{1}{e} < 1$

(d) (2 points) $\sum_{n=1}^{\infty} \frac{4}{n^{\frac{8}{5}}}$. Diverges. P-series with $p \leq 1$.

(e) (2 points) $\sum_{n=1}^{\infty} n(-1.785)^n$. Diverges. Acts like a geometric series, $r = -1.785$, $|r| \geq 1$.

2 (10 points) Determine whether the following series converge or diverge. Justify your answers.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{n^2 + 1}{7n^3 - n \ln n}$. The easiest thing to do here is just use the comparison test to say $\frac{n^2 + 1}{7n^3 - n \ln n} \geq \frac{n^2}{7n^3} = \frac{1}{7n}$, which is a harmonic series, and so diverges. Thus by the comparison test, the original series diverges.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{(2n)^n}{2n^2}$. Since $2^{n^2} = (2^n)^n$, the whole sum is raised to the n^{th} power, so use the root test. This gives $\lim_{n \rightarrow \infty} \left(\frac{(2n)^n}{2n^2}\right)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{2n}{2^n} = \lim_{n \rightarrow \infty} \frac{2}{2^n \ln n} = 0 < 1$, so the series converges by the ratio test.

3 (10 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{\sin n}{n2^n}$. It all goes like this: $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n2^n}\right| \leq \sum_{n=1}^{\infty} \frac{1}{n2^n}$. At this point you can either use the ratio or root test to get the limit is $\frac{1}{2}$, so the series converges absolutely, or you could take the comparison test further, and say $\frac{1}{n2^n} \leq \frac{1}{2^n}$, which as a series is a geometric series which converges. So either way, by the comparison test the series converges absolutely.

(b) (5 points) $\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{3^n (n!)^2}$. Since you see factorials, use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (2(n+1))!}{3^{n+1} ((n+1)!)^2}}{\frac{(-1)^n (2n)!}{3^n (n!)^2}} \right| = \lim_{n \rightarrow \infty} \frac{(2n+2)! (n!)^2}{3((n+1)!)^2 (2n)!} = \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)}{3(n+1)(n+1)} = \frac{4}{3} \geq 1$$
, so the series diverges. Since you obtain divergence through the ratio test, you do not need to test conditional convergence. You're done at this point.

- 4 (10 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} \ln n}$$

First you must check to see if the series converges uniformly. $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n} \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ since for $n \geq 1$, $\sqrt{n} \leq n$. Now, you can deal with this new sum with the integral test, and $\int_2^{\infty} \frac{1}{x \ln x} dx = \ln(\ln n) \Big|_{x=2}^{\infty} = \infty$ so the series diverges. This then shows that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} \ln n}$ diverges, and the original series cannot converge absolutely. But now you must check for conditional convergence, which you do with the alternating series test. Simply state something about the fact that the series is decreasing, and that the limit of the terms is zero as $n \rightarrow \infty$, and you see that the series conditionally converges.

- 5 (10 points) Show all work.

- (a) (5 points) Find $\int_0^x \frac{\ln(1-t)}{t} dt$ using power series. Here you can use the fact that

$$\ln(1-t) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}, \text{ and its radius of convergence is one, to quickly get to the integral:}$$

$$\int_0^x \frac{1}{t} \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} dt = \int_0^x \sum_{n=0}^{\infty} \frac{t^n}{n+1} dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^n}{n+1} dt = \sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2}$$

- (b) (5 points) Find the interval of convergence of the series obtained in part (a). Since the radius of convergence of the series for $\ln(1-x)$ is one, multiplying by a constant ($\frac{1}{x}$) won't change the radius, and neither will integrating it. Or you could confirm this yourself using the ratio test or root test. Either way, once you do this, you must check the endpoints. First check $x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$. In absolute value this is just a p-series with $p=2$, so it converges. If $x = 1$, the series is also a p-series with $p=2$, so it also converges. So this gives that the interval of convergence is $[-2, 2]$.