1 (10 points) Below are five statements. Write clearly whether each is true or false. If a statement is false, say why it is false or give a counterexample.

(a) (2 points) The series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a convergent geometric series.

False. This was the one trick question. This is a convergent p-series.

(b) (2 points) If $\sum |a_n|$ converges, then $\sum a_n$ converges.

True.

(c) (2 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.

False. The harmonic series is the common counterexample here.

(d) (2 points) If both $\sum a_n$ and $\sum b_n$ are divergent, then $\sum(a_n + b_n)$ is divergent.

False. Say you have $\sum \frac{1}{n}$ and $\sum \frac{-1}{n}$, the sum of their terms is zero, but each individually diverges.

(e) (2 points) The alternating harmonic series is a conditionally convergent series.

True

2 (10 points) For the following sequences and series, state whether they converge or diverge. No work is necessary, and no partial credit will be given.

(a) (2 points) $\{\tan n\}_{n=1}^{\infty}$ Diverges. Tangent goes all over the place. Inverse tangent is the one that converges at infinity.

(b) (2 points) $\left\{\frac{n^n}{n!}\right\}_{n=1}^{\infty}$ Diverges. n^n goes to infinity faster than $n!$.

(c) (2 points) $\sum_{n=0}^{\infty}(-\pi)^{-n}$ Converges. This is a geometric series with $r = \frac{-1}{\pi}$.

(d) (2 points) $\sum_{n=1}^{\infty} \frac{1}{n^{0.8}}$ Diverges. This is a p-series with $p = 0.8$.

(e) (2 points) $\sum_{n=1}^{\infty} -e^{-n} \ln n$ Converges. This is essentially a convergent geometric series. Using the ratio test shows that the $\ln n$ doesn't change the fact that it converges.

3 (10 points) Determine whether the following series converge or diverge. Justify your answers.

(a) (5 points) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

Since everything here is raised to a power of n, you want to use the root test.

$$\lim_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$$

This is the indeterminant form 1^{∞}, so you must let $L = \left(\frac{n}{n+1}\right)^n$ and take the natural log of both sides:

$$\ln L = \lim_{n\to\infty} \ln\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right) = \lim_{n\to\infty} \frac{\left(\frac{n}{n+1}\right)}{\frac{1}{n}}.$$

Now use l'Hopital's rule:

$$\lim_{n\to\infty} \frac{\frac{\frac{n+1-n}{(n+1)^2}}{\frac{n}{n+1}}}{\frac{-1}{n^2}} = \lim_{n\to\infty} \frac{-n}{n+1} = -1.$$

So then $\ln L = -1$, or the limit is $e^{-1} < 1$, so by the root test, this series converges.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n + (-1)^n \ln n}{n}$

The easiest way to do this would be to split this into two series. The first is $\sum_{n=1}^{\infty} 1$, which diverges. The second is $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$. This series converges by the alternating series test. A convergesnt series plus a divergent series diverges.

4 (6 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \tan \frac{1}{n}$

First the series must be checked for absolute convergence. As $x \to 0$, $\tan x$ looks like x. So try the limit comparison test: $\displaystyle\lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}}$

As $n \to \infty$, $\frac{1}{n} \to 0$, so this limit is the same as $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$

Therefore by the limit comparison test $\sum_{n=1}^{\infty} \tan \frac{1}{n}$ diverges since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Now for the alternating series test, to see if the series converges conditionally. It has already been seen that $\lim_{n\to\infty} \tan \frac{1}{n} = 0$. Now it just remains to see that it is decreasing. If $f(x) = \tan \frac{1}{x}$, then $f'(x) = \sec^2 \frac{1}{x}(-x^2)$, which is always negative for positive x, so the terms are decreasing. Thus the series converges by the alternating series test, and the series is conditionally convergent.

5 (7 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work. $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^n}{e^{2n} n!}$

Since there's a factorial there's only one thing to do - the ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(-(n+1))^{n+1}}{e^{2(n+1)}(n+1)!}}{\frac{(-n)^n}{e^{2n}n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^{n+1} e^{2n} n!}{n^n e^{2n+2}(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^{n+1} e^{2n}}{n^n e^{2n+2}(n+1)} =$$

$$\lim_{n\to\infty} \frac{(n+1)^n e^{2n}}{n^n e^{2n+2}} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n e^2} = \frac{1}{e^2} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \frac{1}{e^2} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \frac{1}{e^2} e = \frac{1}{e} < 1$$

Thus the series converges absolutely.

6 (7 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n}}$$

For this series, it is fairly obvious by the alternating series test that this series converges. It must be checked whether or not it converges absolutely. In considering the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$, you can either say that the terms in this series satisfy $\frac{1}{n\sqrt{\ln n}} > \frac{1}{n\sqrt{\ln n}}$, which is a series that diverges by the integral test, or you can use the integral test on this series itself.

Begin with the integral: $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\, dx$. Next make the u-substitution $u = \ln x$ to obtain $\int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\, du = \int_{\ln 2}^{\infty} u^{\frac{-1}{2}}\, du = \frac{1}{2} u^{\frac{1}{2}}\Big|_{u=\ln 2}^{\infty} = \infty$. Thus the series doesn't converge absolutely.

Since the series converges but not absolutely, it is a conditionally convergent series.