Below are five statements. Write clearly whether each is true or false. If a statement is false, say why it is false or give a counterexample.

(a) (2 points) The series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent geometric series.
False. This was the one trick question. This is a convergent p-series.

(b) (2 points) If \( \sum |a_n| \) converges, then \( \sum a_n \) converges.
True.

(c) (2 points) If \( \lim_{n \to \infty} a_n = 0 \), then \( \sum a_n \) is convergent.
False. The harmonic series is the common counterexample here.

(d) (2 points) If both \( \sum a_n \) and \( \sum b_n \) are divergent, then \( \sum (a_n + b_n) \) is divergent.
False. Say you have \( \sum \frac{1}{n} \) and \( \sum -\frac{1}{n} \), the sum of their terms is zero, but each individually diverges.

(e) (2 points) The alternating harmonic series is a conditionally convergent series.
True

For the following sequences and series, state whether they converge or diverge. No work is necessary, and no partial credit will be given.

(a) (2 points) \( \{\tan n\}_{n=1}^{\infty} \) Diverges. Tangent goes all over the place. Inverse tangent is the one that converges at infinity.

(b) (2 points) \( \left\{ \frac{n^n}{n!} \right\}_{n=1}^{\infty} \) Diverges. \( n^n \) goes to infinity faster than \( n! \).

(c) (2 points) \( \sum_{n=0}^{\infty} (-\pi)^{-n} \) Converges. This is a geometric series with \( r = \frac{-1}{\pi} \).

(d) (2 points) \( \sum_{n=1}^{\infty} \frac{1}{n^{0.8}} \) Diverges. This is a p-series with \( p = 0.8 \).

(e) (2 points) \( \sum_{n=1}^{\infty} -e^{-n} \ln n \) Converges. This is essentially a convergent geometric series.
Using the ratio test shows that the \( \ln n \) doesn’t change the fact that it converges.

Determine whether the following series converge or diverge. Justify your answers.

(a) (5 points) \( \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2} \)
Since everything here is raised to a power of \( n \), you want to use the root test.

\[
\lim_{n \to \infty} \left( \frac{n}{n+1} \right)^{\frac{1}{n}} = \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n
\]
This is the indeterminant form \( 1^\infty \), so you must let \( L = \left( \frac{n}{n+1} \right)^n \) and take the natural log of both sides:

\[
\ln L = \lim_{n \to \infty} \ln \left( \frac{n}{n+1} \right)^n = \lim_{n \to \infty} n \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \frac{n}{n+1}.
\]
Now use l'Hopital's rule:

\[
\lim_{n \to \infty} \frac{n+1-n}{n+1^2} = \lim_{n \to \infty} \frac{-n}{n+1} = -1.
\]

So then \( \ln L = -1 \), or the limit is \( e^{-1} < 1 \), so by the root test, this series converges.

(b) (5 points) \( \sum_{n=1}^{\infty} \frac{n + (-1)^n \ln n}{n} \)

The easiest way to do this would be to split this into two series. The first is \( \sum_{n=1}^{\infty} \frac{1}{n} \), which diverges. The second is \( \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n} \). This series converges by the alternating series test. A convergent series plus a divergent series diverges.

4 (6 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work. \( \sum_{n=1}^{\infty} \frac{(-1)^n \tan \frac{1}{n}}{n} \)

First the series must be checked for absolute convergence. As \( x \to 0 \), \( \tan x \) looks like \( x \). So try the limit comparison test: \( \lim_{n \to \infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} \)

As \( n \to \infty \), \( \frac{1}{n} \to 0 \), so this limit is the same as \( \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1 \)

Therefore by the limit comparison test \( \sum_{n=1}^{\infty} \tan \frac{1}{n} \) diverges since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

Now for the alternating series test, to see if the series converges conditionally. It has already been seen that \( \lim_{n \to \infty} \tan \frac{1}{n} = 0 \). Now it just remains to see that it is decreasing. If \( f(x) = \tan \frac{1}{x} \), then \( f'(x) = \sec^2 \frac{1}{x} (-x^2) \), which is always negative for positive \( x \), so the terms are decreasing. Thus the series converges by the alternating series test, and the series is conditionally convergent.

5 (7 points) Test the following series for absolute convergence, conditional convergence, or divergence. Show all work. \( \sum_{n=1}^{\infty} \frac{(-n)^n}{e^{2n} n!} \)

Since there's a factorial there's only one thing to do - the ratio test.

\[
\lim_{n \to \infty} \left| \frac{\frac{(-n+1)^{n+1}}{e^{2(n+1)}(n+1)!}}{\frac{(-n)^n}{e^{2n} n!}} \right| = \lim_{n \to \infty} \frac{(n + 1)^n e^{2n} n!}{n e^{2n+2} (n + 1)!} = \lim_{n \to \infty} \frac{(n + 1)^{n+1} e^{2n}}{n e^{2n+2} (n + 1)} = \lim_{n \to \infty} \frac{(n + 1)^{n+1} e^{2n}}{n e^{2n+2}} = \lim_{n \to \infty} \frac{(n + 1)^{n+1} e^{2n}}{n e^{2n+2}} = \lim_{n \to \infty} \frac{(n + 1)^{n+1} e^{2n}}{n e^{2n+2}} = \lim_{n \to \infty} \frac{(n + 1)^{n+1} e^{2n}}{n e^{2n+2}} = \frac{1}{e^2 e} = \frac{1}{e} < 1
\]

Thus the series converges absolutely.
Test the following series for absolute convergence, conditional convergence, or divergence. Show all work.

\[ \sum_{n=2}^{\infty} \frac{(-1)^n}{n^{1/2} \ln n} \]

For this series, it is fairly obvious by the alternating series test that this series converges. It must be checked whether or not it converges absolutely. In considering the series \[ \sum_{n=2}^{\infty} \frac{1}{n^{1/2} \ln n} \],

you can either say that the terms in this series satisfy \[ \frac{1}{n^{1/2} \ln n} > \frac{1}{n^{1/2} \ln n} \], which is a series that diverges by the integral test, or you can use the integral test on this series itself.

Begin with the integral: \[ \int_{2}^{\infty} \frac{1}{x^{1/2} \ln x} \, dx \]. Next make the u-substitution \( u = \ln x \) to obtain

\[ \int_{\ln 2}^{\infty} \frac{1}{u^{1/2}} \, du = \int_{\ln 2}^{\infty} u^{-1/2} \, du = \frac{1}{2} u^{1/2} \bigg|_{u=\ln 2}^{\infty} = \infty. \]

Thus the series doesn’t converge absolutely.

Since the series converges but not absolutely, it is a conditionally convergent series.