Math 126D **Exam Two** **Summer 2003**

Your Name

Your Signature

Student ID #

Problem	Total Points	Score
1	10	
2	10	
3	20	
4	5	
5	5	
Total	50	

- This exam is closed book. You may use one $8\frac{1}{2} \times 11$ sheet of notes.
- Do not share notes.
- Calculators are not allowed.
- In order to receive credit, you must show your work. You must also justify all conclusions you make. Do not assume something is obvious. If you feel something is clear enough to not necessitate algebra, write a sentence or two explaining your reasoning. Do not do computations in your head. Instead, write them out on the exam paper.
- Place a box around $\boxed{\text{YOUR FINAL ANSWER}}$ to each question.
- If you need more room, use the backs of the pages and indicate to me that you have done so.
- Raise your hand if you have a question.

$\boxed{1}$ (10 points) For the following expressions, state whether they yield a vector, a scalar, or doesn't make sense. Here, $\mathbf{a}, \mathbf{b}, \mathbf{c}$,and $\mathbf{d}$ are all vectors. If in doubt, you may assume that none of the quantites involved are zero (or the zero vector). (One point each.)

(a) $\mathbf{a}\cdot\mathbf{b}+\text{comp}_{\mathbf{d}}\mathbf{c}$ This is a scalar plus a scalar, so the answer is $\boxed{\text{scalar}}$.

(b) $\mathbf{a}\times\mathbf{b}+\text{comp}_{\mathbf{d}}\mathbf{c}$ This is a vector plus a scalar, which is impossible, so this $\boxed{\text{Makes no sense}}$.

(c) $\|\mathbf{a}\|\text{proj}_{\mathbf{c}}\mathbf{b}$ This is a scalar multiplied by a vector, which gives a $\boxed{\text{vector}}$.

(d) $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ This is a scalar crossed with a vector, which $\boxed{\text{makes no sense}}$.

(e) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ This is a vector dotted with a vector, so this is a $\boxed{\text{scalar}}$.

(f) $\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ This is an unordered cross product of vectors, which is ambiguous. See page 807 of the book. This $\boxed{\text{makes no sense}}$.

(g) $\mathbf{a}(\mathbf{b}\times\mathbf{c})$ This appears to be a vector multiplied by another vector, but the kind of multiplication is not specified, so this $\boxed{\text{makes no sense}}$.

(h) $\mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ This is a vector multiplied by a scalar, so this is a $\boxed{\text{vector}}$.

(i) $\dfrac{\mathbf{a}\cdot\mathbf{b}}{\text{proj}_{\mathbf{c}}\mathbf{d}}$ This is a scalar divided by a vector. It is impossible to divide by vectors, so this $\boxed{\text{makes no sense}}$.

(j) $\dfrac{\mathbf{a}\times\mathbf{b}}{\text{comp}_{\mathbf{c}}\mathbf{d}}$ This is a vector divided by a scalar, so this gives a $\boxed{\text{vector}}$.

$\boxed{2}$ (10 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}(x+3)^n.$$

I prefer using the ratio test, though the root test works just as well. This series will converge when

$$\lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}}{\frac{(-2)^n(x+3)^n}{\sqrt{n}}}\right|<1$$

This limit simplifies to:

$$\lim_{n\to\infty}2|x+3|\sqrt{\frac{n}{n+1}}=2|x+3|$$

So the series converges when $2|x+3|<1$, or $|x+3|<\frac{1}{2}$, or $-\frac{1}{2}<x+3<\frac{1}{2}$, or when $-\frac{7}{2}<x<-\frac{5}{2}$. Now the endpoints must be checked. If $x=-\frac{7}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}(-\frac{1}{2})^n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which is a divergent p-series, so the power series diverges at this endpoint. If $x=-\frac{5}{2}$, the series becomes $\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}(\frac{1}{2})^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which is a convergent alternating series. This gives that the interval of convergence is $\boxed{-\frac{7}{2}<x\le-\frac{5}{2}}$ or $\boxed{(-\frac{7}{2},-\frac{5}{2}]}$.

$\boxed{3}$ (20 points) Consider the points

$$P = (1, -1, 2), Q = (3, 1, 2), R = (-2, -2, 1), \text{ and } S = (1, 0, 1)$$

.

(a) (14 points) Find the distance from the point P to the plane containing $Q, R,$ and S.
First you need information about the plane containing Q, R, and S. So you need the normal vector to the plane and a point on the plane. Thankfully there are plenty of points on the plane already known, so you just need to worry about the normal line. The normal vector is perpendicular to all vectors in the plane, so find to distinct vectors and cross them to get the vector perpendicular to the both of them. I choose the vectors $\overrightarrow{QR}$ and $\overrightarrow{QS}$, though others are just as good.
$\overrightarrow{QR} = < -5, -3, -1 >$, and $\overrightarrow{QS} = < -2, -1, -1 >$. So the normal vector, $\vec{n}$, is

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -3 & -1 \\ -2 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} -5 & -1 \\ -2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} -5 & -3 \\ -2 & -1 \end{vmatrix} \vec{k}$$

$$= (3-1)\vec{i} - (5-2)\vec{j} + (5-6)\vec{k} = 2\vec{i} - 3\vec{j} - \vec{k} = < 2, -3, -1 >$$

The actual equation of the plane is then (using this normal vector and the point S) $2(x-1) - 3y - (z-1) = 0$ or $2x - 3y - z - 1 = 0$.
So the distance from the point to the plane is: $\dfrac{|2 - 3(-1) - 1(2) - 1|}{\sqrt{4+9+1}} = \boxed{\frac{2}{\sqrt{14}}}$.

(b) (6 points) Find the volume of the parallelepiped spanned by $\overrightarrow{PQ}$, $\overrightarrow{PR}$, and $\overrightarrow{PS}$.
$\overrightarrow{PQ} = < 2, 2, 0 >$, $\overrightarrow{PR} = < -3, -1, -1 >$, and$\overrightarrow{PS} = < 0, 1, -1 >$. The volume of a parallelepiped spanned by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $|\vec{a} \cdot (\vec{b} \times \vec{c})$.

$$\overrightarrow{PQ} \cdot (\overrightarrow{PR} \times \overrightarrow{PS}) = \begin{vmatrix} 2 & 2 & 0 \\ -3 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = 2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} -3 & -1 \\ 0 & -1 \end{vmatrix} + 0 \begin{vmatrix} -3 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 2(1+1) - 2(3+0) + 0 = 4 - 6 = -2$$

But the area is the absolute value of this, so the area of the parallelepipe is $|-2| = \boxed{2}$.

$\boxed{4}$ (5 points) Find the symmetric equation of the line contained in the two planes

$$\begin{aligned} 2x + y - 3z + 2 &= 0 \\ -x - y + 2z - 4 &= 0 \end{aligned}$$

The line is contained in both planes, so the normal vector of each plane must be perpendicular to the vector in the direction of the line. So the direction vector for the line is the cross product of the normal vectors of the two planes.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \vec{k}$$

$$= (2-3)\vec{i} - (4-3)\vec{j} + (-2+1)\vec{k} = -\vec{i} - \vec{j} - \vec{k} =< -1, -1, -1 >$$

Note that any (nonzero) scalar multiple of this vector is equally valid (e.g. $< 1, 1, 1 >$). Now all that is needed for the line is a point on the line. In other words, some point that satisfies both plane equations at the same time. Since there are two equations but three unknowns, one of the variables can be chosen freely and then the other two can be found from there.

I picked $z = 0$. This then gives the two remaining equations as

$$\begin{aligned} 2x + y + 2 &= 0 \\ -x - y - 4 &= 0 \end{aligned}$$

Add these two equations to get that $x - 2 = 0$, or $x = 2$, plug this in to one of the two equations to get $y = -6$. So the point obtained is $(2, -6, 0)$. Putting all this information into the symmetric equation of a line yields $\boxed{\frac{x-2}{-1} = \frac{y+6}{-1} = \frac{z}{-1}}$.

$\boxed{5}$ (5 points) Using the dot product, show that the three points $P = (2, 1, 1)$, $Q = (5, -1, 2)$, and $R = (3, 2, 0)$ form a right triangle.

Examine the three vectors formed between the points of the triangle: $\overrightarrow{PQ}=< 3, -2, 1 >$, $\overrightarrow{PR}=< 1, 1, -1 >$, and $\overrightarrow{QR}=< -2, 3, -2 >$.

$\overrightarrow{PQ} \cdot \overrightarrow{PR}= 3(1) - 2(1) + 1(-1) = 0$. Thus $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are perpendicular, and the triangle is a right triangle.